Performance of HSAGE Method with Seikkala Derivative for 2-D Fuzzy Poisson equation

A. A. Dahalan, J. Sulaiman
School of Science and Technology
Universiti Malaysia Sabah
88400 Kota Kinabalu, Sabah, Malaysia

M. S. Muthuvalu
Faculty of Science and Information Technology
Universiti Teknologi PETRONAS
31750 Tronoh, Perak, Malaysia

Copyright © 2014 A. A. Dahalan, J. Sulaiman and M. S. Muthuvalu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this study, iterative methods particularly families of Alternating Group Explicit (AGE) methods are used to solve system of linear equations generated from the discretization of two-point fuzzy boundary value problems (FBVPs). The formulation and implementation of the Full-Sweep AGE (FSAGE) and Half-Sweep AGE (HSAGE) methods were also presented. Then numerical experiments are carried out onto two example problems to verify the effectiveness of the methods.

Keywords: finite difference scheme, fuzzy boundary value problems, complexity reduction approach

1 Introduction

Fuzzy boundary value problems (FBVPs) and treating fuzzy differential equations were one of the major applications for fuzzy number arithmetic [5]. FBVPs can be approached by two types. For instance, the first approach addresses problems in
which the boundary values are fuzzy where the solution is still in fuzzy function. Then the second approach is based on generating the fuzzy solution from the crisp solution [13]. To solve these problems, numerical methods obtain their approximate solution. Consequently, in this paper, consider the two dimensional fuzzy Poisson equation (2DFPE) be given as

\[
\begin{align*}
\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} &= f(x, y) \\
U(x, 0) &= g_1(x) \\
U(x, a) &= g_2(x) \quad 0 \leq x \leq a, \\
U(0, y) &= \tilde{g}_1(y) \\
U(a, y) &= \tilde{g}_2(y) \quad 0 \leq y \leq a,
\end{align*}
\]

where \( f(x, y), g_1(x), g_2(x), \tilde{g}_1(y) \) and \( \tilde{g}_2(y) \) were fuzzy numbers or fuzzy functions. Actually there exist various derivatives of fuzzy function such as Goetschel-Voxman, Buckley-Feuring, Dubois-Prade, Puri-Ralescu and Kandel-Friedman-Ming [5]. In fact, Bede and Hukuhara derivative [2,4,16,17] is one of the usual methods used in solving fuzzy boundary problems. Since the Seikkala derivative is one of the most general solutions to the fuzzy initial value problem [21], this paper deals with the application of the Alternating Group Explicit (AGE) iterative method in solving linear systems generated from finite difference approximation equations.

In order to solve the 2DFPE (1) numerically based on the Seikkala derivative, we apply the second-order central finite difference scheme to discretize the 2DFPE into linear systems. Basically, many iterative methods [7,14,20,23] can be used to solve the linear system. However in this paper, the generated linear systems will be solved iteratively by using AGE method [8,10]. Indeed, this iterative method is also known as the two stage iterative method, which is suitable for parallel computation in solving the associated matrix equation. Again AGE method, which is also analogous to Alternate Direction Implicit (ADI) scheme has been used extensively in solving large scale computations. From previous studies, findings of the papers related to the AGE iterative method and its variants [11,18,22,24] have shown that the efficiency of the family of AGE methods has been widely used to solve the non-fuzzy problems. Due to the efficiency of the methods, this paper extends the application of AGE iterative method in solving fuzzy problems.

Since the fuzzy linear systems will be constructed, the iterative method becomes the natural option to get a fuzzy numerical solution of the problem. Based on the previous work of the family of iterative methods, the discovery on the concept of the half-sweep iterative method has been inspired by Abdullah [1] via Explicit Decoupled Group (EDG) iterative method in solving two-dimensional
Poisson equations. The main characteristic of this concept is that the half-sweep iterative method includes the reduction technique in order to reduce the computational complexity of linear systems generated from corresponding approximation equations. To simplify the formulation of the full- and half-sweep central difference approximation equations, the finite grid network will be used as shown in Fig. 1. Implementations of these proposed iterative methods are executed onto the interior solid nodal points until convergence test is found. Meanwhile, the approximation solutions for the remaining points can be computed by using direct method [1,6,15].

Fig. 1 a) and b) show distribution of uniform solid node points for the full- and half-sweep cases respectively

2 Finite Difference Approximation Equations

As mentioned in the first section, the main objective of this paper is to get the finite difference solution of two dimensional fuzzy Poisson equation. To do this, let $\tilde{x}$ and $\tilde{y}$ be two fuzzy subset of real numbers. They are characterized by the corresponds membership function evaluated at $t$, written $\tilde{x}(t)$ and $\tilde{y}(t)$ respectively as a number in $[0,1]$. A $\alpha$ - cut of $\tilde{x}$ and $\tilde{y}$, which $\alpha$ is denoted as a crisp number, can be written as $\tilde{x}(\alpha)$ and $\tilde{y}(\alpha)$ in $\{x|\tilde{x}(t) \geq \alpha\}$ and $\{y|\tilde{y}(t) \geq \alpha\}$, for $0 < \alpha \leq 1$. The interval of the $\alpha$ - cut of fuzzy numbers will be written as $\tilde{x}(\alpha)=[\underline{x}\alpha, \overline{x}\alpha]$ and $\tilde{y}(\alpha)=[\underline{y}\alpha, \overline{y}\alpha]$, for all $\alpha$, since they were always closed and bounded [3]. Suppose $\{x, \tilde{x}\}$ and $\{y, \tilde{y}\}$ be parametric form of fuzzy function $x$ and $y$ respectively, now for arbitrary positive integer $n$ subdivide the interval $a \leq t \leq b$ whereas $t_i = a + ih$ ($i = 0,1,2,...,n$) and $t_j = a + jh$ ($j = 0,1,2,...,n$)
for $i$ and $j$ respectively and $h = \frac{b-a}{n}$.

Denote the value of $x$ and $y$ as $(x_i, \tilde{x})$ and $(y_j, \tilde{y})$ at the representative point $t_i, (i = 0,1,2,\ldots,n)$ and $t_j, (j = 0,1,2,\ldots,n)$ by $x_i$ and $y_j$ at $(x_i, x_i)$ and $(y_j, y_j)$ respectively. Thus, by using the second-order central finite difference scheme, for problem (1) can be developed as

\[
\frac{\partial^2 U}{\partial x^2} \bigg|_{i,j} = \frac{U_{i+p,j} - 2U_{i,j} + U_{i-p,j}}{(ph)^2}, \quad (2)
\]

and

\[
\frac{\partial^2 U}{\partial y^2} \bigg|_{i,j} = \frac{U_{i,j+p} - 2U_{i,j} + U_{i,j-p}}{(ph)^2}, \quad (3)
\]

give

\[
\frac{\partial^2 U}{\partial x^2} \bigg|_{i,j} = \left( \frac{\partial^2 U}{\partial x^2}, \frac{\partial^2 U}{\partial y^2} \right), \quad (6)
\]

and

\[
\frac{\partial^2 U}{\partial y^2} \bigg|_{i,j} = \left( \frac{\partial^2 U}{\partial x^2}, \frac{\partial^2 U}{\partial y^2} \right). \quad (7)
\]

Let

\[
\tilde{U}(x, y) = U(x, y), \tilde{U}(x, y) = F(x, y).
\]

Then, by using parametric form of fuzzy function, (1) can be written as

\[
\frac{\partial^2 U}{\partial x^2}(x, y) + \frac{\partial^2 U}{\partial y^2}(x, y) = f(x, y), \quad (9)
\]

and

\[
\frac{\partial^2 U}{\partial x^2}(x, y) + \frac{\partial^2 U}{\partial y^2}(x, y) = \tilde{f}(x, y). \quad (10)
\]

For full-sweep cases, by applying (2) and (4), (9) will be reduces to

\[
4U_{i,j} = \left( \tilde{U}_{i+1,j} + \tilde{U}_{i-1,j} \right) + \left( \tilde{U}_{i,j+1} + \tilde{U}_{i,j-1} \right) - h^2 \tilde{f}_{i,j}, \quad (11)
\]

for $i, j = 1,2,\ldots,n-1$. Meanwhile, by substituting (3) and (5) into (10), we will have

\[
4\tilde{U}_{i,j} = \left( \tilde{U}_{i+1,j} + \tilde{U}_{i-1,j} \right) + \left( \tilde{U}_{i,j+1} + \tilde{U}_{i,j-1} \right) - h^2 \tilde{f}_{i,j}., \quad (12)
\]

Whereas for half-sweep cases could be written as

\[
4U_{i,j} = \left( \tilde{U}_{i-1,j+1} + \tilde{U}_{i+1,j-1} \right) + \left( \tilde{U}_{i,j+1} + \tilde{U}_{i,j-1} \right) - 2h^2 \tilde{f}_{i,j}. \quad (13)
\]

And
Performance of HSAGE method with Seikkala derivative

\[ 4\overline{U}_{i,j} = (\overline{U}_{i-1,j} + \overline{U}_{i+1,j}) + (\overline{U}_{i,j-1} + \overline{U}_{i,j+1}) - 2h^2\overline{f}_{i,j} \]  \hspace{1cm} (14)

respectively for \( i, j = 1,2,\ldots,n-1 \). Since equation (11) until (14) have the same form in terms of the equation, except, based on the interval of the \( \alpha \)-cuts, the differences identified only in the upper and lower bound, thus it can be rewritten as

\[ 4U_{i,j} = (U_{i,j+1} + U_{i,j-1}) + (U_{i+1,j} + U_{i-1,j}) - h^2f_{i,j} \]  \hspace{1cm} (15)

for full-sweep and

\[ 4U_{i,j} = (U_{i,j-1} + U_{i,j+1}) + U_{i,j} - 2h^2f_{i,j} \]  \hspace{1cm} (16)

for half-sweep cases.

Now, we can express the second-order central finite difference approximations (15) and (16) in matrix form as

\[ A\tilde{U} = \tilde{F} \]  \hspace{1cm} (17)

with

\[ A = \begin{bmatrix} p_0 & p_1 & & & \\ p_2 & 0 & p_1 & & \\ & \ddots & \ddots & \ddots & \\ & & p_2 & 0 & p_1 \\ & & & p_2 & 0 \\ & & & & p_0 \end{bmatrix} \quad \text{and} \quad \tilde{F} = \begin{bmatrix} I & p_{01} & & & \\ & I & p_{01} & & \\ & & \ddots & \ddots & \ddots \\ & & & I & p_{01} \\ & & & & I \end{bmatrix} (n-1) \times (n-1) \]

\[ p_{01} = \frac{1}{15} \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}, \quad p_{02} = \frac{1}{15} \begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix} \]

According to linear system (17), \( \tilde{U} \) and \( \tilde{F} \) are unknown and known vectors respectively.

### 3 Alternating Group Explicit Iterative Method

Consider a class of methods be mentioned in [9] which is based on the splitting of the matrix \( A \) into the sum of its constituent symmetric and positive definite matrices, as follows

\[ A = G_1 + G_2 + G_3 + G_4 \]  \hspace{1cm} (19)

where \( G_1 \) and \( G_2 \) are the forward and backward differences in the \( x \)-plane and \( G_3 \) and \( G_4 \) are similar difference in \( y \)-plane. Then \( \text{diag}(G_1) = \text{diag}(G_2) = \frac{1}{4} \text{diag}(A) \) with
By reordering the points column-wise along $y$-direction, $G_1$ and $G_4$ literally have the same structure as $G_3$ and $G_2$ respectively,

$$
\tilde{G}_3 = G_1 = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & \cdots & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & \cdots & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1
\end{bmatrix}_{n^2 \times n^2}.
$$
and

\[
\begin{bmatrix}
1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & -1 & 1 \\
\end{bmatrix}
\]

\[
G_4 = G_2 =
\begin{bmatrix}
1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & -1 & 1 \\
\end{bmatrix}
\]

Then (17) becomes

\[
(G_1 + G_2 + G_3 + G_4)U = F
\]

Thus, the explicit form of AGE method can be written as

\[
U^{(k+\frac{1}{2})} = (r_1 I + G_1)^{-1}\left[ 2f + (r_1 I + G_1 - 2A) \right],
\]

(21)

\[
U^{(k+\frac{1}{2})} = (r_1 I + G_1)^{-1}\left[ G_2 U^{(k)} + r_1 U^{(k+\frac{1}{2})} \right],
\]

(22)

\[
U^{(k+\frac{1}{2})} = (r_2 I + G_2)^{-1}\left[ G_3 U^{(k)} + r_2 U^{(k+\frac{1}{2})} \right],
\]

(23)

and

\[
U^{(k+\frac{1}{2})} = (r_1 I + G_1)^{-1}\left[ G_4 U^{(k)} + r_2 U^{(k+\frac{1}{2})} \right].
\]

(24)

From (21) until (24), therefore, the implementation of the families of AGE methods is presented in Algorithm 1.
Algorithm 1: Families of AGE methods

i. Initialize $U^{(0)} \leftarrow 0$ and $\varepsilon \leftarrow 10^{-10}$.

ii. First sweep
Compute
$$U^{(k+\frac{1}{2})} = (r_1 I + G_1)^{\dagger}\left[2 f + (r_1 I + G_1 - 2A)\right]$$

iii. Second sweep
Compute
$$U^{(k+\frac{1}{2})} = (r_1 I + G_2)^{\dagger}\left[G_2 U^{(k)} + r_1 U^{(k+\frac{1}{2})}\right]$$

iv. Third sweep
Compute
$$U^{(k+1)} = (r_2 I + G_3)^{\dagger}\left[G_3 U^{(k)} + r_2 U^{(k+\frac{1}{2})}\right]$$

v. Fourth sweep
Compute
$$U^{(k+1)} = (r_3 I + G_4)^{\dagger}\left[G_4 U^{(k)} + r_3 U^{(k+\frac{1}{2})}\right]$$

vi. Convergence test. If the convergence criterion i.e. $\lim_{k \to \infty} \|U^{(k+1)} - U^{(k)}\| \leq \varepsilon$ is satisfied, go to Step (vii). Otherwise go back to Step (ii).

vii. Display approximate solutions.

4 Numerical Experiments

Two examples of 2DFPE are considered to verify the effectiveness of FSGS, FSAGE and HSAGE methods. For comparison purposes, three parameters were observed that are number of iterations, execution time (in seconds) and Hausdorff distance (as mentioned in Definition 2).

**Definition 1** [19]

Given two minimum bounding rectangles P and Q, a lower bound of the Hausdorff distance from the elements confined by P to the elements confined by Q is defined as

$$HausDistLB(P,Q) = \text{Max}[\text{MinDist}(f_a, Q): f_a \in \text{FacesOf}(P)]$$

**Problem 1** [3]

$$\frac{\partial^2 U}{\partial x^2}(x, y) + \frac{\partial^2 U}{\partial y^2}(x, y) = \tilde{k}xe^x, \quad 0 \leq x \leq 2, \ 0 \leq y \leq 1$$  \hspace{1cm} (25)

where $\tilde{k} = [\tilde{k}(\alpha)] = [0.75 + 0.25\alpha, 1.25 - 0.25\alpha]$ with the boundary conditions
\( U(0, y) = 0, \quad \bar{U}(2, y) = 2\kappa e^y, \quad 0 \leq y \leq 1 \quad \text{and} \quad \bar{U}(x, 0) = \bar{k}x, \quad \bar{U}(x, 1) = \bar{k}e^x, \quad 0 \leq x \leq 2 \). The exact solution for

\[
\frac{\partial^2 U}{\partial x^2}(x, y; \alpha) + \frac{\partial^2 U}{\partial y^2}(x, y; \alpha) = k(\alpha)xe^y
\]

and

\[
\frac{\partial^2 \bar{U}}{\partial x^2}(x, y; \alpha) + \frac{\partial^2 \bar{U}}{\partial y^2}(x, y; \alpha) = \bar{k}(\alpha)xe^y
\]

are

\( U(x, y; \alpha) = k(\alpha)xe^y \) \hspace{1cm} (28)

and

\( \bar{U}(x, y; \alpha) = \bar{k}(\alpha)xe^y \) \hspace{1cm} (29)

respectively.

**Problem 2** [1]

\[
\frac{\partial^2 \bar{U}}{\partial x^2}(x, y) + \frac{\partial^2 \bar{U}}{\partial y^2}(x, y) = k\left(x^2 + y^2\right)e^{\gamma y}, \quad 0 < x < 2, \quad 0 < y < 1
\]

(30)

where \( k[\alpha] = [k(\alpha), \bar{k}(\alpha)] = [0.75 + 0.25\alpha, 1.25 - 0.25\alpha] \) with the boundary conditions \( \bar{U}(0, y) = 0, \quad \bar{U}(2, y) = 2\kappa e^y, \quad 0 \leq y \leq 1 \quad \text{and} \quad \bar{U}(x, 0) = \bar{k}x, \quad \bar{U}(x, 1) = \bar{k}e^x, \quad 0 \leq x \leq 2 \). The exact solution for

\[
\frac{\partial^2 U}{\partial x^2}(x, y; \alpha) + \frac{\partial^2 U}{\partial y^2}(x, y; \alpha) = k(\alpha)(x^2 + y^2)e^{\gamma y}
\]

and

\[
\frac{\partial^2 \bar{U}}{\partial x^2}(x, y; \alpha) + \frac{\partial^2 \bar{U}}{\partial y^2}(x, y; \alpha) = \bar{k}(\alpha)(x^2 + y^2)e^{\gamma y}
\]

are

\( U(x, y; \alpha) = k(\alpha)e^{\gamma y} \) \hspace{1cm} (33)

and

\( \bar{U}(x, y; \alpha) = \bar{k}(\alpha)e^{\gamma y} \) \hspace{1cm} (34)

respectively.

Based on these two problems, numerical results for FSGS, FSAGE and HSAGE methods have been recorded in Tables 1 to 5.

### 5 Conclusions

In this paper, the families of AGE methods were used to solve linear systems arisen from the discretization of two-point FBVPs using the second-order central finite difference scheme. The results showed that HSAGE method is more superior in terms of the number of iterations, execution time and Hausdorff distance compared to the FSAGE and FSGS methods. Since AGE is well suited for parallel computation, it can be considered as a main advantage because this
method has groups of independent task which can be implemented simultaneously. It is hoped that the capability of the proposed method will be helpful for the further investigation in solving any multi-dimensional fuzzy partial differential equations [12]. Also the family of AGE methods such as as Modified Alternating Group Explicit (MAGE) [11,24] and Two Parameter Alternating Group Explicit (TAGE) [18,22] methods can be used as linear solvers in solving the same problem. Basically the results of this paper can be classified as one of full- and half-sweep iterations.

References


TABLE 1. Comparison of three parameters between FSGS, FSAGE and HSAGE methods at $\alpha = 0.00$.

<table>
<thead>
<tr>
<th>Problem 1</th>
<th>Problem 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>FSGS</td>
</tr>
<tr>
<td>32</td>
<td>3883</td>
</tr>
<tr>
<td>64</td>
<td>14366</td>
</tr>
<tr>
<td>128</td>
<td>52818</td>
</tr>
<tr>
<td>256</td>
<td>192760</td>
</tr>
<tr>
<td>512</td>
<td>697178</td>
</tr>
</tbody>
</table>

Execution Time

<table>
<thead>
<tr>
<th>$n$</th>
<th>FSGS</th>
<th>FSAGE</th>
<th>HSAGE</th>
<th>FSGS</th>
<th>FSAGE</th>
<th>HSAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>0.10</td>
<td>0.08</td>
<td>0.25</td>
<td>0.11</td>
<td>0.07</td>
<td>0.23</td>
</tr>
<tr>
<td>64</td>
<td>1.52</td>
<td>1.17</td>
<td>0.61</td>
<td>1.57</td>
<td>1.19</td>
<td>0.60</td>
</tr>
<tr>
<td>128</td>
<td>23.05</td>
<td>17.62</td>
<td>6.05</td>
<td>23.44</td>
<td>18.11</td>
<td>6.18</td>
</tr>
<tr>
<td>256</td>
<td>337.26</td>
<td>273.52</td>
<td>81.35</td>
<td>344.97</td>
<td>281.43</td>
<td>83.18</td>
</tr>
<tr>
<td>512</td>
<td>5231.84</td>
<td>5445.99</td>
<td>1890.16</td>
<td>5333.03</td>
<td>5682.34</td>
<td>1915.02</td>
</tr>
</tbody>
</table>

Hausdorff Distance

<table>
<thead>
<tr>
<th>$n$</th>
<th>FSGS</th>
<th>FSAGE</th>
<th>HSAGE</th>
<th>FSGS</th>
<th>FSAGE</th>
<th>HSAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>6.831e-06</td>
<td>6.8446e-06</td>
<td>6.8631e-06</td>
<td>3.8277e-06</td>
<td>3.8320e-06</td>
<td>2.0275e-04</td>
</tr>
<tr>
<td>64</td>
<td>1.6773e-06</td>
<td>1.7042e-06</td>
<td>1.8228e-06</td>
<td>9.3734e-07</td>
<td>9.5464e-07</td>
<td>5.0694e-05</td>
</tr>
<tr>
<td>128</td>
<td>2.927e-07</td>
<td>3.9027e-07</td>
<td>4.0928e-07</td>
<td>1.6698e-07</td>
<td>2.1685e-07</td>
<td>1.2651e-05</td>
</tr>
<tr>
<td>512</td>
<td>2.6413e-06</td>
<td>6.8798e-07</td>
<td>3.3781e-07</td>
<td>2.6498e-06</td>
<td>6.9641e-07</td>
<td>5.1598e-07</td>
</tr>
</tbody>
</table>

TABLE 2. Comparison of three parameters between FSGS, FSAGE and HSAGE methods at $\alpha = 0.25$.

<table>
<thead>
<tr>
<th>Problem 1</th>
<th>Problem 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>FSGS</td>
</tr>
<tr>
<td>32</td>
<td>3886</td>
</tr>
<tr>
<td>64</td>
<td>14379</td>
</tr>
<tr>
<td>128</td>
<td>52866</td>
</tr>
<tr>
<td>256</td>
<td>192951</td>
</tr>
<tr>
<td>512</td>
<td>697942</td>
</tr>
</tbody>
</table>

Execution Time

<table>
<thead>
<tr>
<th>$n$</th>
<th>FSGS</th>
<th>FSAGE</th>
<th>HSAGE</th>
<th>FSGS</th>
<th>FSAGE</th>
<th>HSAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>0.10</td>
<td>0.08</td>
<td>0.24</td>
<td>0.10</td>
<td>0.08</td>
<td>0.19</td>
</tr>
<tr>
<td>64</td>
<td>1.53</td>
<td>1.16</td>
<td>0.60</td>
<td>1.56</td>
<td>1.20</td>
<td>0.61</td>
</tr>
<tr>
<td>128</td>
<td>22.97</td>
<td>17.64</td>
<td>6.09</td>
<td>23.52</td>
<td>18.08</td>
<td>6.22</td>
</tr>
<tr>
<td>256</td>
<td>337.57</td>
<td>274.09</td>
<td>81.63</td>
<td>345.29</td>
<td>281.21</td>
<td>83.01</td>
</tr>
<tr>
<td>512</td>
<td>5167.50</td>
<td>5443.12</td>
<td>1911.61</td>
<td>5341.94</td>
<td>5700.13</td>
<td>1925.45</td>
</tr>
</tbody>
</table>

Hausdorff Distance

<table>
<thead>
<tr>
<th>$n$</th>
<th>FSGS</th>
<th>FSAGE</th>
<th>HSAGE</th>
<th>FSGS</th>
<th>FSAGE</th>
<th>HSAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>1.5916e-06</td>
<td>1.6185e-06</td>
<td>1.7314e-06</td>
<td>8.8939e-07</td>
<td>9.0663e-07</td>
<td>4.8159e-05</td>
</tr>
<tr>
<td>256</td>
<td>6.0056e-07</td>
<td>1.1383e-07</td>
<td>4.0952e-08</td>
<td>6.3716e-07</td>
<td>1.4960e-07</td>
<td>2.9316e-06</td>
</tr>
<tr>
<td>512</td>
<td>2.6401e-06</td>
<td>6.8671e-07</td>
<td>3.3661e-07</td>
<td>2.6492e-06</td>
<td>6.9595e-07</td>
<td>4.7887e-07</td>
</tr>
</tbody>
</table>
### TABLE 3. Comparison of three parameters between FSGS, FSAGE and HSAGE methods at $\alpha = 0.50$.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Number of iterations</th>
<th>Execution Time</th>
<th>Hausdorff Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FSGS</td>
<td>FSAGE</td>
<td>HSAGE</td>
</tr>
<tr>
<td>$n$ 32</td>
<td>3888</td>
<td>1109</td>
<td>578</td>
</tr>
<tr>
<td>64</td>
<td>14387</td>
<td>4106</td>
<td>2015</td>
</tr>
<tr>
<td>128</td>
<td>52899</td>
<td>15172</td>
<td>7919</td>
</tr>
<tr>
<td>256</td>
<td>193084</td>
<td>55768</td>
<td>29199</td>
</tr>
<tr>
<td>512</td>
<td>698475</td>
<td>203507</td>
<td>106972</td>
</tr>
</tbody>
</table>

### TABLE 4. Comparison of three parameters between FSGS, FSAGE and HSAGE methods at $\alpha = 0.75$.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Number of iterations</th>
<th>Execution Time</th>
<th>Hausdorff Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FSGS</td>
<td>FSAGE</td>
<td>HSAGE</td>
</tr>
<tr>
<td>$n$ 32</td>
<td>3889</td>
<td>1109</td>
<td>578</td>
</tr>
<tr>
<td>64</td>
<td>14392</td>
<td>4107</td>
<td>2015</td>
</tr>
<tr>
<td>128</td>
<td>52920</td>
<td>15177</td>
<td>7922</td>
</tr>
<tr>
<td>256</td>
<td>193163</td>
<td>55788</td>
<td>29210</td>
</tr>
<tr>
<td>512</td>
<td>698789</td>
<td>203590</td>
<td>107013</td>
</tr>
</tbody>
</table>
TABLE 5. Comparison of three parameters between FSGS, FSAGE and HSAGE methods at $\alpha = 1.00$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Problem 1</th>
<th></th>
<th></th>
<th>Problem 2</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FSGS</td>
<td>FSAGE</td>
<td>HSAGE</td>
<td>FSGS</td>
<td>FSAGE</td>
<td>HSAGE</td>
</tr>
<tr>
<td>32</td>
<td>3890</td>
<td>1110</td>
<td>578</td>
<td>3972</td>
<td>1130</td>
<td>590</td>
</tr>
<tr>
<td>64</td>
<td>14394</td>
<td>4108</td>
<td>2016</td>
<td>14736</td>
<td>4198</td>
<td>2184</td>
</tr>
<tr>
<td>128</td>
<td>52926</td>
<td>15178</td>
<td>7922</td>
<td>54328</td>
<td>15550</td>
<td>8108</td>
</tr>
<tr>
<td>256</td>
<td>193188</td>
<td>55796</td>
<td>29212</td>
<td>198866</td>
<td>57296</td>
<td>29966</td>
</tr>
<tr>
<td>512</td>
<td>698892</td>
<td>203618</td>
<td>107026</td>
<td>721750</td>
<td>209662</td>
<td>110060</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n$</th>
<th>Problem 1</th>
<th></th>
<th></th>
<th>Problem 2</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FSGS</td>
<td>FSAGE</td>
<td>HSAGE</td>
<td>FSGS</td>
<td>FSAGE</td>
<td>HSAGE</td>
</tr>
<tr>
<td>32</td>
<td>0.10</td>
<td>0.08</td>
<td>0.23</td>
<td>0.10</td>
<td>0.08</td>
<td>0.22</td>
</tr>
<tr>
<td>64</td>
<td>1.54</td>
<td>1.16</td>
<td>0.56</td>
<td>1.56</td>
<td>1.19</td>
<td>0.63</td>
</tr>
<tr>
<td>128</td>
<td>23.00</td>
<td>17.70</td>
<td>6.00</td>
<td>23.46</td>
<td>18.13</td>
<td>6.18</td>
</tr>
<tr>
<td>256</td>
<td>335.82</td>
<td>274.50</td>
<td>81.76</td>
<td>345.64</td>
<td>281.83</td>
<td>83.38</td>
</tr>
<tr>
<td>512</td>
<td>5234.67</td>
<td>5524.80</td>
<td>1899.91</td>
<td>5347.16</td>
<td>5694.80</td>
<td>1961.01</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n$</th>
<th>Problem 1</th>
<th></th>
<th></th>
<th>Problem 2</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FSGS</td>
<td>FSAGE</td>
<td>HSAGE</td>
<td>FSGS</td>
<td>FSAGE</td>
<td>HSAGE</td>
</tr>
<tr>
<td>32</td>
<td>5.4688e-06</td>
<td>5.4753e-06</td>
<td>5.4903e-06</td>
<td>3.0611e-06</td>
<td>3.0653e-06</td>
<td>1.6220e-04</td>
</tr>
<tr>
<td>64</td>
<td>1.3345e-06</td>
<td>1.3615e-06</td>
<td>1.4574e-06</td>
<td>7.4552e-07</td>
<td>7.6248e-07</td>
<td>4.0554e-05</td>
</tr>
<tr>
<td>256</td>
<td>5.8616e-07</td>
<td>1.0031e-07</td>
<td>2.8910e-08</td>
<td>6.3112e-07</td>
<td>1.4399e-07</td>
<td>2.4566e-06</td>
</tr>
<tr>
<td>512</td>
<td>2.6364e-06</td>
<td>6.8306e-07</td>
<td>3.3298e-07</td>
<td>2.6477e-06</td>
<td>6.9442e-07</td>
<td>3.6968e-07</td>
</tr>
</tbody>
</table>

Received: November 14, 2013