Formula for the Number of Spanning Trees in Light Graph

Hajar Sahbani and Mohamed El Marraki

LRIT, Associated Unit to CNRST (URAC No 29)
Mohammed V-Agdal University, B.P.1014 RP, Agdal, Morocco

Abstract

In this paper, we consider the outerplanar graph $L_n[1]$, having $6n+6$ vertices, $12n+9$ edges and $6n+5$ faces, in this graph all faces have degree 3 except for the outside face.

Our approach consists on finding a general formula that calculates the number of spanning trees in the Ligth graph $L_n$, depending on $n$.

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1 Introduction

The number of spanning trees in a planar graph (network) is an important well-studied quantity and invariant of the graph; moreover it is also an important measure of reliability of a network which plays a central role in Kirchhoff’s classical theory of electrical networks. In a graph (network), that contains several cycles, we must remove the redundancies in this network, i.e., we obtain a spanning tree. A spanning tree in a planar graph $G$ is a tree which has the same vertex set as $G$ (tree that passing through all the vertices of the map $G$)[2].

Generally, the number of spanning trees in a network can be obtained by computing a related determinant of the Laplacian matrix defined by $L(G) =$
$D(G) - A(G)$, with $D(G)$ and $A(G)$ are respectively the matrix of degrees and the adjacency matrix [3, 4]. However, for a large graph, evaluating the relevant determinant is computationally intractable. Wherefore, many works derive formulas to calculate the complexity for some classes of graphs. Bogdanowicz [7] derive the explicit formula $\tau(F_n)$; the number of spanning trees in $F_n$. A. Modabish and M. El Marraki investigated the number of spanning trees in the star flower planar graph [8]. In [9] the authors proposed an approach for counting the number of spanning trees in the butterfly graph.

In the following, we describe a general method to count the number of spanning trees in the outerplanar light graph $L_n$ [1], our work consist on combining several method as presented in [5], in order to calculate the complexity of $L_n$.

## 2 Preliminary Notes

An undirected graph is outerplanar if it can be drawn in the plane without crossings such that all vertices lie on the outerface boundary. That is, no vertex is totally surrounded by edges.

Let $G_n$ the set of outerplanar graph shown in Figure 1. where $v_1, s_1$ and $v_2$ are the vertices of the outer face boundary of a plane network that delimit a number $n$ of vertices which is the same between each pair of $(v_1, s_1)$ and $(s_1, v_2)$, also the total number of vertices of $G_n$ is $|V_{G_n}| = 2n + 3$ and $n \geq 1$.

![Figure 1: The family of graphs $G_n$](image)

The connection of three $G_n$ graphs lead to the outerplanar light graph shown in Figure 2.
Property 2.1 The number of vertices, edges and faces of light graph satisfy respectively: $|V_{L_n}| = |V_{L_{n-1}}| + 6 = 6(n + 1)$, $|E_{L_n}| = |E_{L_{n-1}}| + 12 = 3(3 + 4n)$ and $|F_{L_n}| = |F_{L_{n-1}}| + 6 = 6n + 5$.

Proof: The number of vertices of light graph is calculated recursively as:

$$|V_{L_n}| = |V_{L_{n-1}}| + 6 = \ldots = |V_{L_0}| + 6n = 6(n + 1).$$

The same for edges and faces.

Before presenting the main results we need the following results:

Let $G = G_1 \bullet G_2$ obtained by connecting $G_1$ and $G_2$ with a single vertex $v_1$ [5], then

$$\tau(G_1 \bullet G_2) = \tau(G_1) \times \tau(G_2).$$

(1)

Let $G$ be a planar graph of type $G = G_1 | G_2$, ($v_1$ and $v_2$ two vertices of $G_1$ and $G_2$ connected by an edge $e$) [5], then

$$\tau(G) = \tau(G_1) \times \tau(G_2) - \tau(G_1 - e) \times \tau(G_2 - e).$$

(2)

Theorem 2.2 [5] Let $G = G_1 : G_2$ be a planar graph, $v_1$ and $v_2$ two vertices of $G$ which is formed by two planar graphs $G_1$ and $G_2$, then

$$\tau(G) = \tau(G_1) \times \tau(G_2.v_1v_2) + \tau(G_1.v_1v_2) \times \tau(G_2).$$

Theorem 2.3 [5, 6, 7] The number of spanning trees of the Fan ($F_n$) shown in the left of Figure 3, with $(n = |V_{F_n}| - 2)$ satisfies:

$$\tau(F_n) = 3\tau(F_{n-1}) - \tau(F_{n-2})$$

$$= \frac{1}{\sqrt{5}} \left( \left( \frac{3 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{3 - \sqrt{5}}{2} \right)^{n+1} \right), \quad n \geq 1$$
3 Main Results

In this section we present some results to calculate the number of spanning trees in some families of outerplanar graphs to evaluate the complexity of light graph \((L_n)\).

3.1 The complexity of \(X_n\) graph

\(X_n\) is the outerplanar graph illustrated on the right of Figure 3, having one vertex of degree \(n\), 1 of degree 4, 2 vertices of degree 2 and the rest of degree 3, where \(|V_{X_n}| = n + 2\).

\[\text{Figure 3: } F_n \text{ and } X_n \text{ graphs}\]

Lemma 3.1 The number of spanning trees of \(X_n\) is equal to the number of spanning trees of the \(n\)-fan:

\[\tau(X_n) = \frac{1}{\sqrt{5}}((\frac{3 + \sqrt{5}}{2})^{n+1} - (\frac{3 - \sqrt{5}}{2})^{n+1}), \ n \geq 1\]

Proof: we apply equation (2)

\[\text{Figure 4: The complexity of } X_n \text{ according to equation (2)}\]

then:

\[\tau(X_n) = 3\tau(F_{n-1}) - \tau(F_{n-2}) = \tau(F_n)\]
3.2 The complexity of $G_n$ graph

$G_n$ is the outerplanar graph represented in Figure 1, with $|V_{G_n}| = 2n + 3$.

**Lemma 3.2** The number of spanning trees of the $G_n$ graph depends on $\tau(F_n)$:

$$\tau(G_n) = 2\tau(F_n)\left(5\tau(F_{n+1}) - 12\tau(F_n)\right)$$

**Proof:** By using equation (2) we get

Figure 5: The complexity of $G_n$ according to equation (2)

with:

\[
\begin{align*}
\tau(X_n) & = \tau(F_n) \quad \text{from Lemma 3.1} \\
\tau(I_n) & = 3\tau(F_{n-2}) \quad \text{from equation (1)}
\end{align*}
\]

then:

$$\tau(G_n) = \tau(F_n)\left(\tau(F_{n+1}) - 3\tau(F_{n-2})\right)$$ (3)

other hand:

$$\tau(F_{n+1}) = 3\tau(F_n) - \tau(F_{n-1}) \implies \tau(F_{n-1}) = 3\tau(F_n) - \tau(F_{n+1})$$ (4)

and:

$$\tau(F_{n-2}) = 3\tau(F_{n-1}) - \tau(F_n) \implies \tau(F_{n-2}) = 8\tau(F_n) - 3\tau(F_{n+1})$$ (5)

we replace $\tau(F_{n-2})$ of equation (5) in (3), then the result.

**Corollary 3.3** The complexity of the outerplanar graph $G_n$ is given by the following formula:

$$\tau(G_n) = \left(\frac{6 + 4\sqrt{5}}{5}\right)\left(\frac{7 + 3\sqrt{5}}{2}\right)^n + \left(\frac{6 - 4\sqrt{5}}{5}\right)\left(\frac{7 - 3\sqrt{5}}{2}\right)^n + \frac{18}{5}, \ n \geq 1$$

**Proof:** We use formula of Theorem 2.3 depending on $n$ in Lemma 3.2, then the result.

3.3 The complexity of $H_n$ graph

$H_n$ is the outerplanar graph presented bellow in Figure 6:a, with $n = \frac{|V_{H_n}| - 2}{2}$.
**Lemma 3.4** The complexity of $H_n$ depends on $G_n$ and $F_n$:

$$\tau(H_n) = \tau(G_n) - \tau(F_n)^2$$

**Proof:** From equation (2):

$$\tau(H_n) = \tau(F_n)\left(2\tau(F_n) - \tau(F_{n-1}) - 3\tau(F_{n-2})\right)$$  \hspace{1cm} (6)

we replace $\tau(F_{n-1})$ and $\tau(F_{n-2})$ in (6) using equations (4) and (5) of proof of Lemma 3.2, so:

$$\tau(H_n) = \tau(F_n)\left(10\tau(F_{n+1}) - 25\tau(F_n)\right)$$

with: $\tau(G_n) = \tau(F_n)\left(10\tau(F_{n+1}) - 24\tau(F_n)\right)$, from Lemma 3.2.

**Corollary 3.5** The complexity of the $H_n$ graph is given by the following formula:

$$\tau(H_n) = \left(\frac{1 + \sqrt{5}}{2}\right)\left(\frac{7 + 3\sqrt{5}}{2}\right)^n + \left(\frac{1 - \sqrt{5}}{2}\right)\left(\frac{7 - 3\sqrt{5}}{2}\right)^n + 4, \; n \geq 1$$
3.4 The complexity of $A_n$ graph

$A_n$ is the outerplanar graph illustrated in Figure 8:a, with $|V_{A_n}| = 4n + 4$.

**Lemma 3.6** The number of spanning trees of $A_n$ is calculated by:

$$
\tau(A_n) = 2\tau(G_n) \left( \tau(G_n) - \tau(F_n)^2 \right)
$$

$$
= -\left( \frac{102 + 46\sqrt{5}}{25} \right) \left( \frac{7 + 3\sqrt{5}}{2} \right)^{2n} - \left( \frac{102 - 46\sqrt{5}}{25} \right) \left( \frac{7 - 3\sqrt{5}}{2} \right)^{2n} + \left( \frac{24 - 16\sqrt{5}}{25} \right) \left( \frac{3 - \sqrt{5}}{2} \right)^{2n}
$$

$$
+ \left( \frac{60 + 50\sqrt{5}}{5} \right) \left( \frac{7 + 3\sqrt{5}}{2} \right)^n + \left( \frac{306 - 234\sqrt{5}}{25} \right) \left( \frac{7 - 3\sqrt{5}}{2} \right)^n + \left( \frac{232 + 96\sqrt{5}}{25} \right) \left( \frac{47 + 21\sqrt{5}}{2} \right)^n
$$

$$
+ \left( \frac{232 - 96\sqrt{5}}{25} \right) \left( \frac{47 - 21\sqrt{5}}{2} \right)^n + \frac{116}{5}, \quad n \geq 1
$$

**Proof:** Again equation (2) implies that

$$
\tau(A_n) = \tau(N_n) \times \tau(G_n) - \tau(G_n) \times \tau(O_n)
$$

with:

$$
\begin{align*}
\tau(N_n) &= 2\tau(G_n) - \tau(F_n)^2 \\
\tau(O_n) &= \tau(F_n)^2
\end{align*}
$$

then the result.
3.5 The complexity of light graph $L_n$

Results found previously allow us to calculate the number of spanning trees of light graph $L_n$: $n = \frac{|V_{L_n}| - 6}{6}$.

![Figure 10: a: Light graph $L_n$ and b: the $M_n$ graph](image)

**Theorem 3.7** The number of spanning trees in $L_n$ is given by the following formula:

$$
\tau(L_n) = 3\tau(G_n)^2 \left( \tau(G_n) - \tau(F_n)^2 \right)
$$

$$
= \left( \frac{5508 + 2484\sqrt{5}}{125} \right) \left( \frac{7 + 3\sqrt{5}}{2} \right) 2^n - \left( \frac{5508 - 2484\sqrt{5}}{125} \right) \left( \frac{7 - 3\sqrt{5}}{2} \right) 2^n
$$

$$
+ \left( \frac{1296 - 864\sqrt{5}}{125} \right) \left( \frac{3 - \sqrt{5}}{2} \right) 2^n + \left( \frac{12876 + 9504\sqrt{5}}{125} \right) \left( \frac{7 + 3\sqrt{5}}{2} \right) n
$$

$$
+ \left( \frac{2316 - 1728\sqrt{5}}{125} \right) \left( \frac{7 - 3\sqrt{5}}{2} \right)^n + \left( \frac{19488 + 8064\sqrt{5}}{125} \right) \left( \frac{47 + 21\sqrt{5}}{2} \right)^n
$$

$$
+ \left( \frac{19488 - 8064\sqrt{5}}{125} \right) \left( \frac{47 - 21\sqrt{5}}{2} \right)^n + \left( \frac{534 + 246\sqrt{5}}{25} \right) \left( \frac{161 + 72\sqrt{5}}{2} \right)^n
$$

$$
+ \left( \frac{534 - 246\sqrt{5}}{25} \right) \left( \frac{161 - 72\sqrt{5}}{2} \right)^n + \left( \frac{924 + 396\sqrt{5}}{125} \right) \left( \frac{7 + 3\sqrt{5}}{2} \right)^n \left( \frac{7 - 3\sqrt{5}}{2} \right)^n
$$

$$
+ \left( \frac{924 - 396\sqrt{5}}{125} \right) \left( \frac{7 - 3\sqrt{5}}{2} \right)^2 \left( \frac{7 + 3\sqrt{5}}{2} \right)^n + \frac{264}{5}, \ n \geq 1
$$

**Proof:** We cut $L_n$ as shown in Figure 11.
The two subgraphs have vertices $v_1$ and $v_2$ in common, so using Theorem 2.2 we get

\[ \tau(L_n) = \tau(M_n) \times \tau(H_n) + \tau(G_n) \times \tau(A_n) \]

with: \( \tau(M_n) = \tau(G_n)^2 \) from equation (1), and using Lemma 3.6 and 3.4 we obtain the result.

References


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