Symmetric Identities for the Generalized Tangent Polynomials Associated with $p$-Adic Integral on $\mathbb{Z}_p$

C. S. Ryoo

Department of Mathematics
Hannam University, Daejeon 306-791, Korea

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Abstract

In this paper, we study the symmetry for the generalized tangent numbers $T_{n,\chi}$ and polynomials $T_{n,\chi}(x)$. We obtain some interesting identities of the power sums and the generalized polynomials $T_{n,\chi}(x)$ using the symmetric properties for the $p$-adic invariant integral on $\mathbb{Z}_p$.

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1 Introduction

Euler numbers, Euler polynomials, and tangent numbers, tangent polynomials were studied by many authors. Bernoulli numbers, Bernoulli polynomials, Euler numbers, Euler polynomials, tangent numbers, and tangent polynomials possess many interesting properties and arise in many areas of mathematics and physics. These numbers are still in the center of the advanced mathematical research(see [1-8]). Especially, in number theory and quantum theory, they have many applications. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of $p$-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers,
Note that $p_{\text{fermionic}}$ with $d$ be the space of uniformly differentiable function on $\mathbb{Z}$ assume that $(\text{see [2]})$

Follows: (\text{cf. [7]}) introduced the generalized tangent numbers $C \in \mathbb{Q}$ defined by the generating function:

$$T \equiv \text{Ryoo [7] introduced the generalized tangent numbers } C \in \mathbb{Q} \text{ defined by the generating function:}$$

If we take $g_n(x) = g(x + n)$ in (1.1), then we see that

$$I_{-1}(g_n) = (-1)^n I_{-1}(g) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} g(l). \quad (1.2)$$

Let a fixed positive integer $d$ with $(p, d) = 1$, set

$$X = X_d = \lim_{N \to \infty} \mathbb{Z}/dp^N \mathbb{Z}, X_1 = \mathbb{Z}_p, X_1 = \bigcup_{0 < a < dp^N, (a, p) = 1} a + dp \mathbb{Z}_p,$$

$$a + dp^N \mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^N} \},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$. It is easy to see that

$$I_{-1}(g) = \int_X g(x) d\mu_{-1}(x) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x), \text{ see [2].} \quad (1.3)$$

Ryoo [7] introduced the generalized tangent numbers $T_{n, \chi}$ and polynomials $T_{n, \chi}(x)$ attached to $\chi$. Let $\chi$ be Dirichlet’s character with conductor $d \in \mathbb{N}$ with $d \equiv 1 (\text{mod } 2)$. The generalized tangent numbers $T_{n, \chi}$ attached to $\chi$ are defined by the generating function:

$$2 \sum_{a=0}^{d-1} \chi(a)(-1)^a e^{2at} \equiv \sum_{n=0}^{\infty} T_{n, \chi} \frac{t^n}{n!}; \text{ cf. [7].} \quad (1.4)$$

We consider the generalized tangent polynomials $T_{n, \chi}(x)$ attached to $\chi$ as follows:

$$\left( 2 \sum_{a=0}^{d-1} \chi(a)(-1)^a e^{2at} \equiv \sum_{n=0}^{\infty} T_{n, \chi} \frac{t^n}{n!} \right) e^{xt} = \sum_{n=0}^{\infty} T_{n, \chi}(x) \frac{t^n}{n!}. \quad (1.5)$$
Theorem 1.1 For positive integers \( n \), we have
\[
T_{n,\chi}(x) = \int_X \chi(y)(2y + x)^n d\mu_1(y).
\]

Theorem 1.2 For positive integers \( n \), we have
\[
T_{n,\chi}(x) = \sum_{l=0}^{n} \binom{n}{l} T_{l,\chi} x^{n-l}.
\]

2 Symmetry for the generalized tangent polynomials

In this section, by using same method of [2], except for obvious modifications, we obtain recurrence identities the generalized tangent polynomials \( T_{n,\chi}(x) \) and the alternating sums of powers of consecutive integers. If \( n \) is odd from (1.2), we obtain
\[
I_{-1}(g_n) + I_{-1}(g) = 2 \sum_{k=0}^{n-1} (-1)^k g(k) \quad (\text{see [2], [3]}).
\]

Substituting \( g(x) = \chi(x)e^{2xt} \) into the above, we obtain
\[
\int_X \chi(x + n)e^{(2x+2n)t} d\mu_1(x) + \int_X \chi(x)e^{2xt} d\mu_1(x) = 2 \sum_{j=0}^{n-1} (-1)^j \chi(j)e^{2jt}.
\]

For \( k \in \mathbb{N} \cup \{0\} \), let us define the power sums \( T_{k,\chi}(n) \) as follows:
\[
T_{k,\chi}(n) = \sum_{l=0}^{n} (-1)^l \chi(l)(2l)^k.
\]

After some elementary calculations, we have
\[
\int_X \chi(x)e^{(2x+2nd)t} d\mu_1(x) + \int_X \chi(x)e^{2xt} d\mu_1(x) = 2 \sum_{a=0}^{d-1} \chi(a)(-1)^a e^{2at}
\frac{e^{2dt} + 1}{e^{2dt} + 1}.
\]

From the above, we get
\[
\int_X \chi(x)e^{(2x+2nd)t} d\mu_1(x) + \int_X \chi(x)e^{2xt} d\mu_1(x)
\frac{2 \int_X \chi(x)e^{2xt} d\mu_1(x)}{\int_X e^{2ndtx} d\mu_1(x)}.
\]
By substituting Taylor series of $e^{2xt}$ into (2.2), we obtain
\[
\sum_{m=0}^{\infty} \left( \int_X \chi(x)(2x + 2nd)^m d\mu(x) + \int_X \chi(x)(2x)^m d\mu_1(x) \right) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left( 2 \sum_{j=0}^{nd-1} (-1)^j \chi(j)(2j)^m \right) \frac{t^m}{m!}
\] (2.5)

By comparing coefficients $\frac{t^m}{m!}$ in the above equation and (2.3), we obtain
\[
\sum_{k=0}^{m} \binom{m}{k} (2nd)^{m-k} \int_X \chi(x)(2x)^k d\mu_1(x) + \int_X \chi(x)(2x)^m d\mu_1(x) = 2T_{m,\chi}(nd-1).
\] (2.6)

By using (2.4) and (2.6), we arrive at the following theorem:

**Theorem 2.1** Let $n$ be odd positive integer. Then we obtain
\[
\frac{2 \int_X \chi(x)e^{2xt} d\mu_1(x)}{\int_X e^{2ndx} d\mu_1(x)} = \sum_{m=0}^{\infty} \left( 2T_{m,\chi}(nd-1) \right) \frac{t^m}{m!}.
\]

Let $w_1$ and $w_2$ be odd positive integers. Then we set
\[
S(w_1, w_2) = \frac{\int_X \chi(x_1)\chi(x_2) e^{(2w_1x_1+2w_2x_2+w_1w_2)x} d\mu_1(x_1) d\mu_1(x_2)}{\int_X e^{2w_1w_2dx} d\mu_1(x)}
\] (2.7)

By Theorem 2.1 and (2.7), after elementary calculations, we obtain
\[
S(w_1, w_2) = \left( \frac{1}{2} \int_X \chi(x_1) e^{(2w_1x_1+w_1w_2)x} d\mu_1(x_1) \right) \frac{2 \int_X \chi(x_2) e^{2xw_2x} d\mu_1(x_2)}{\int_X e^{2w_1w_2dx} d\mu_1(x)}
\] = \left( \frac{1}{2} \sum_{m=0}^{\infty} T_{m,\chi}(w_2x)w_1^m \frac{t^m}{m!} \right) \left( 2 \sum_{m=0}^{\infty} T_{m,\chi}(w_1d-1)w_2^m \frac{t^m}{m!} \right).

By using Cauchy product in the above, we have
\[
S(w_1, w_2) = \sum_{m=0}^{\infty} \left( \sum_{j=0}^{m} \binom{m}{j} T_{j,\chi}(w_2x)w_1^j \frac{t^m}{m!} \right) \left( 2 \sum_{m=0}^{\infty} T_{m,\chi}(w_1d-1)w_2^m \frac{t^m}{m!} \right).
\] (2.8)

From the symmetry of $S(w_1, w_2)$ in $w_1$ and $w_2$, we also see that
\[
S(w_1, w_2) = \left( \frac{1}{2} \int_X \chi(x_2) e^{(2w_2x_2+w_1w_2)x} d\mu_1(x_2) \right) \frac{2 \int_X \chi(x_1) e^{2xw_1x} d\mu_1(x_1)}{\int_X e^{2w_1w_2dx} d\mu_1(x)}
\] = \left( \frac{1}{2} \sum_{m=0}^{\infty} T_{m,\chi}(w_1x)w_2^m \frac{t^m}{m!} \right) \left( 2 \sum_{m=0}^{\infty} T_{m,\chi}(w_2d-1)w_1^m \frac{t^m}{m!} \right).
Thus we have

\[ S(w_1, w_2) = \sum_{m=0}^{\infty} \left( \sum_{j=0}^{m} \binom{m}{j} T_{j,\chi}(w_1) w_1^m T_{m-j,\chi}(w_2) \right) \frac{t^m}{m!} \quad (2.9) \]

By comparing coefficients \( \frac{t^m}{m!} \) in the both sides of (2.8) and (2.9), we arrive at the following theorem:

**Theorem 2.2** Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we obtain

\[
\sum_{j=0}^{m} \binom{m}{j} w_1^{m-j} w_2^j T_{j,\chi}(w_1) T_{m-j,\chi}(w_2 d - 1)
\]

\[
= \sum_{j=0}^{m} \binom{m}{j} w_1^{j} w_2^{m-j} T_{j,\chi}(w_2 x) T_{m-j,\chi}(w_1 d - 1),
\]

where \( T_{k,\chi}(x) \) and \( T_{m,\chi}(k) \) denote the generalized tangent polynomials and the alternating sums of powers, respectively.

By Theorem 1.2, we have the following corollary.

**Corollary 2.3** Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we obtain

\[
\sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j} \binom{j}{k} w_1^{m-k} w_2^j x^{j-k} T_{k,\chi} w_2 T_{m-j,\chi}(w_2 d - 1)
\]

\[
= \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j} \binom{j}{k} w_1^{j} w_2^{m-k} x^{j-k} T_{k,\chi} T_{m-j,\chi}(w_1 d - 1).
\]

Now we will derive another interesting identities for the generalized tangent polynomials using the symmetric property of \( S(w_1, w_2) \).

\[
S(w_1, w_2) = \left( \frac{1}{2} \int_X \chi(x) e^{(2w_1x_1 + w_1w_2x) t} d\mu_{-1}(x_1) \right) \left( 2 \int_X \chi(x) e^{2x_2w_2 t} d\mu_{-1}(x_2) \right)
\]

\[
= \left( \frac{1}{2} \int_X \chi(x) e^{2x_1w_1t} d\mu_{-1}(x_1) \right) \left( 2 \sum_{j=0}^{w_1d-1} (-1)^j \chi(j) e^{2jw_2 t} \right)
\]

\[
= \sum_{j=0}^{w_1d-1} (-1)^j \chi(j) \int_X \chi(x) e^{\left( 2x_1 + w_2 x + \frac{2jw_2}{w_1} \right)(w_1 t)} d\mu_{-1}(x_1)
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{j=0}^{w_1d-1} (-1)^j \chi(j) T_{n,\chi} \left( w_2 x + \frac{2jw_2}{w_1} \right) \right) \frac{t^n}{n!},
\]

(2.10)
By using the symmetry property in (2.10), we also have
\[
S(w_1, w_2) = \left(\frac{1}{2} e^{w_1 w_2 t} \int_X \chi(x_2) e^{2x_2 w_2 t} d\mu_{-1}(x_2)\right) \left(\frac{2 \int_X \chi(x_1) e^{2x_1 w_1 t} d\mu_{-1}(x_1)}{\int_X e^{2w_1 w_2 x} d\mu_{-1}(x)}\right)
\]
\[
= \left(\frac{1}{2} e^{w_1 w_2 t} \int_X \chi(x_2) e^{2x_2 w_2 t} d\mu_{-1}(x_2)\right) \left(\frac{2 \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) e^{2j w_1 t}}{\int_X \chi(x_2) e^{2x_2 w_2 t} d\mu_{-1}(x_2)}\right)
\]
\[
= \sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) \int_X \chi(x_2) e^{2x_2 w_2 t} d\mu_{-1}(x_2)
\]
\[
= \sum_{n=0}^{w_2 d-1} \left(\sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) T_{n,\chi} \left(w_1 x + \frac{2j w_1}{w_2}\right) w_2^n\right) \frac{t^n}{n!}.
\]

(2.11)

By comparing coefficients \( \frac{t^n}{n!} \) in the both sides of (2.10) and (2.11), we have the following theorem.

**Theorem 2.4** Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we obtain

\[
\sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) T_{n,\chi} \left(w_2 x + \frac{2j w_2}{w_1}\right) w_1^n
\]

\[
= \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) T_{n,\chi} \left(w_1 x + \frac{2j w_1}{w_2}\right) w_2^n.
\]

If we take \( x = 0 \) in Theorem 2.4, we also derive the interesting identity for the generalized tangent numbers as follows:

\[
\sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) T_{n,\chi} \left(\frac{2j w_2}{w_1}\right) w_1^n = \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) T_{n,\chi} \left(\frac{2j w_1}{w_2}\right) w_2^n.
\]

**References**


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