Modified Multi-Step Iterative Processes
for a Finite Family of Asymptotically
Pseudocontractive Maps

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Abstract

Very recently, Mogbademu [6] answered in affirmative the open problem proposed in [9] by extending an iterative algorithm for two asymptotically pseudocontractive mappings to three mappings. He further asked if his results could be extended to the case of finite family of mappings. This sequel gives a positive answer to the open problem and equally extends so many results in this area of research.

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1 Introduction

The study of fixed point theory was first introduced by Banach in 1922 to solve problem of nonlinear functional equations. Since its introduction, many researchers have been investigating fixed points of nonlinear operators, for which
the Mann, Ishikawa, Noor and multi-step iteration schemes have been prominent. Picard iteration which is also known as repeated iteration or method of successive approximations was developed by Picard in 1890 to establish the existence of solutions of initial value problems for ordinary differential equations. But where this iteration procedure fails to converge some other iteration procedures are introduced. For instance, Mann\([4]\) iteration (one-step) scheme was introduced in 1953 and was used to obtain convergence of a fixed point for many functions for which picard iteration fails to converge.

In 1974 Ishikawa introduced another iteration scheme known as Ishikawa\([3]\) (two-step) iteration to establish convergence for Lipschitzian pseudocontractive maps. Other iterations were introduced where these iterations fails to converge or solve one problem or the other. For example, Noor\([8]\) iteration (three-step) was introduced to solve problems of variational inequality.

In 2004, Rhoades and Soltuz \([10]\) introduced the multi-step iteration. These iteration schemes have been modified and applied to approximate the fixed points of various classes of pseudocontractive maps (See, \([5]\)). These include the class of asymptotically pseudocontractive maps introduced by Schu \([11]\).

Not too long, Rafiq et al. \([9]\) proved a result on the iterative algorithm for two asymptotically pseudocontractive mappings. They gave an open problem which was answered in affirmative by one of the authors of this paper \([6]\) extending their results to three asymptotically pseudocontractive mappings. The purpose of this sequel is to address the open problem of \([6]\), extending their results to multi-step asymptotically pseudocontractive mappings.

\section{Preliminary}

Let \(X\) be a real Banach space and let \(X^*\) be its dual space. The normalized duality mapping \(J : X \rightarrow 2^{X^*}\) is defined by

\[J(x) = \{f \in X^* : \langle x, f \rangle = \|x\|\|f\|\},\]

where \(\langle \cdot, \cdot \rangle\) denotes the generalized duality pairing and \(\|x\| = \|f\|.\) We shall denote the single-value duality pairing by \(j.\)

Consistent with Goebel and Kirk \([2]\) who introduced the class of asymptotically nonexpansive mappings we have the following definitions.

Let \(K\) be a nonempty closed convex subset of \(E\) and \(T : K \rightarrow K\) be a map.

\textbf{Definition 2.1.} A mapping \(T\) is said to be asymptotically nonexpansive if
for each $x, y \in K$

$$< T^n x - T^n y > \leq k_n \| x - y \|$$

$\forall n \geq 0$, where the sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$.

**Definition 2.2.** The mapping is said to be uniformly $L$-Lipschitzian if there exists a constant $L \geq 0$ such that

$$\| T^n x - T^n y \| \leq L \| x - y \|,$$

for any $x, y \in K$ and $\forall n \geq 0$.

We give the definition of asymptotically pseudocontractive maps as in [11].

**Definition 2.3.** The mapping $T$ is said to be asymptotically pseudocontractive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ and for any $x, y \in K$ there exist $j(x - y) \in J(x - y)$ such that

$$< T^n x - T^n y, j(x - y) > \leq k_n \| x - y \|^2, \forall n \geq 0.$$

**Remark 2.4.** Every asymptotically nonexpansive mapping is asymptotically pseudocontractive. Clearly, every operator which is asymptotically pseudocontractive in general may not admit a fixed point. The existence of fixed point result for asymptotically pseudocontractive maps depend on the space, nature of subset and further properties of the operators (See, [1]).

Recently, Chang et al., [1] extended the results in [3] to uniformly smooth Banach space and applied the recursion formula of Ishikawa iterative processes. They proved the following theorem:

**Theorem 2.5.** [1]. Let $E$ be a real Banach space , $K$ be a nonempty closed convex subset of $E$, $T_i : K \to K, (i = 1, 2)$ be two uniformly $L_i$-Lipschitzian mappings with $F(T_1) \cap F(T_2) \neq \phi$, where $F(T_i)$ is the set of fixed points of $T_i$ in $K$ and $\rho$ be a point in $F(T_1) \cap F(T_2)$. Let $\{k_n\} \subset [1, \infty)$ be a sequence with $k_n \to 1$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two sequences in $[0,1]$ satisfying the following:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ (ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ (iii) $\sum_{n=1}^{\infty} \beta_n < \infty$

(iv) $\sum_{n=1}^{\infty} \alpha_n (k_n - 1) < \infty$.

For any $x_1 \in K$, let $\{x_n\}_{n=1}^{\infty}$ be the iterative sequence defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha T_1^n y_n$$

$$y_n = (1 - \beta_n)x_n + \beta T_2^n x_n, \ n \geq 1.$$
If there exists a strictly increasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T_i^nx - \rho, j(x - \rho) > \leq k_n \|x - \rho\|^2 - \Phi(\|x - \rho\|)$$

for all $j(x - \rho) \in J(x - \rho)$ and $x \in K, (i = 1, 2)$, then $\{x_n\}_{n=1}^\infty$ converges strongly to $\rho$.

In 2009, Rafiq, Acu and Sofonea [9] improved the results of Chang et al. [1] by removing the condition $\sum_{n=1}^\infty \alpha_n^2 < \infty$ and $\sum_{n=1}^\infty \alpha_n(k_n - 1) < \infty$ from the Theorem 2.5. Then, they gave an open problem whether their results can be extended for the case of mappings which are more general than the two maps. They proved the following theorem:

**Theorem 2.6.** [6] Let $K$ be a nonempty closed convex subset of a real Banach space $E$, $T_i : K \rightarrow K, (i = 1, 2)$ be two uniformly $L_i$-Lipschitzian mappings with sequence $\{k_n\} \subset [1, \infty), \sum_{n=1}^\infty \alpha_n(k_n - 1) < \infty$ such that $F(T_1) \cap F(T_2) \neq \phi$, where $F(T_i)$ is the set of fixed points of $T_i$ in $K$ and $\rho$ be a point in $F(T_1) \cap F(T_2)$. Let $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ be two sequences in $[0,1]$ such that $\sum_{n=1}^\infty \alpha_n = \infty, \lim_{n \rightarrow \infty} = \beta_n = 0$. For any $x_0 \in K$, let $\{x_n\}_{n=1}^\infty$ be the iterative sequence defined by

$$X_{n+1} = (1 - \alpha_n)x_n + \alpha T_1^ny_n$$

$$y_n = (1 - \beta_n)x_n + \beta T_2^nx_n$$

(2.1)

If there exists a strictly increasing function $\Phi : [0, \infty) \rightarrow [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T_i^nx - \rho, j(x - \rho) > \leq k_n \|x - \rho\|^2 - \Phi(\|x - \rho\|)$$

$\forall j(x - \rho) \in J(x - \rho)$ and $x \in K, (i = 1, 2)$. Then $\{x_n\}_{n=1}^\infty$ converges strongly to $\rho \in F(T_1) \cap F(T_2)$.

Very recently, the second author of this paper [6] addressed the open problem of [9] and gave a positive answer. He extended their result to the case of three maps which are more general than two. Infact, he proved the Theorem below:

**Theorem 2.7** [6]. Let $K$ be a nonempty closed convex subset of a real Banach space $E$, $T_i : K \rightarrow K, (i = 1, 2, 3)$ be three uniformly $L_i$-Lipschitzian mappings
with sequence \( \{k_n\} \subset [1, \infty), \sum_{n=1}^{\infty} (k_n - 1) < \infty \) such that
\[ F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset, \] where \( F(T_i) \) is the set of fixed points of \( T_i \) in \( K \) and \( \rho \) be a point in \( F(T_1) \cap F(T_2) \cap F(T_3) \). Let \( \{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \) and \( \{\gamma_n\}_{n=1}^{\infty} \) be three sequences in \([0,1]\) such that \( \sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n \to \infty} \beta_n = \gamma_n = 0 \). For any \( x_1 \in K \), let \( \{x_n\}_{n=1}^{\infty} \) be the iterative sequence defined by
\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha T_1^n y_n \]
\[ y_n = (1 - \beta_n)x_n + \beta T_2^n z_n \]
\[ z_n = (1 - \gamma_n)x_n + \gamma T_3^n x_n, \quad n \geq 1 \quad (2.2) \]
If there exists a strictly increasing function \( \Phi : [0, \infty) \to [0, \infty) \) with \( \Phi(0) = 0 \) such that
\[ < T_i^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|) \]
\( \forall j(x - \rho) \in J(x - \rho) \) and \( x \in K, (i = 1, 2, 3) \). Then \( \{x_n\}_{n=1}^{\infty} \) converges strongly to \( \rho \in F(T_1) \cap F(T_2) \cap F(T_3) \).

Mogbademu [6] gave an open problem whether his result can be extended to the finite family of maps which are more general than the three maps. The purpose of this paper is to find an answer to the open problem of [6] by following their line of argument. The following concepts and lemmas will be useful.

Definition 2.8[11]. Let \( T_1, T_2, \ldots, T_i : K \to K \) be finite family of maps. For any given \( x_1 \in K \), the multi-step iteration \( \{x_n\}_{n=0}^{\infty} \subset K \) is defined by
\[ x_{n+1} = (1 - b_n)x_n + b_n T_1^n y_n, \quad n \geq 1 \]
\[ y_n^i = (1 - b_n^i)x_n + b_n^i T_i^n y_n^i, \quad i = 1, 2, \ldots, p - 2 \]
\[ y_n^{p-1} = (1 - b_n^{p-1})x_n + b_n^{p-1} T_p^n x_n, \quad p \geq 2. \quad (2.3) \]

Remark 2.9. If \( p = 2,3 \) in (2.3) the two steps iteration (2.1) and the three steps iteration (2.2) are obtained respectively.

Lemma 2.10[9]. Let \( J : E \to 2^{E^*} \) be the normalized duality mapping. Then for any \( x, y \in E \), we have
\[ \|x + y\|^2 \leq \|x\|^2 + 2 < y, j(x + y) >, \forall j(x + y) \in J(x, y). \]
Lemma 2.11[7]. Let $\Phi : [0, \infty) \to [0, \infty)$ be an increasing function with $\Phi(x) = 0 \iff x = 0$ and let $\{e_n\}_{n=0}^{\infty}$ be a positive real sequence satisfying $\sum_{n=0}^{\infty} e_n = +\infty$ and $\lim_{n \to \infty} e_n = 0$. Suppose that $\{d_n\}_{n=0}^{\infty}$ is a nonnegative real sequence. If there exists an integer $N_0 > 0$ satisfying

$$d_{n+1}^2 < d_n^2 + \phi(e_n) - e_n \Phi(d_{n+1}), \forall n \geq N_0$$

where $\lim_{n \to \infty} \frac{\phi(e_n)}{e_n} = 0$, then $\lim_{n \to \infty} d_n = 0$.

3 Main Results

Theorem 3.1 Let $K$ be a nonempty closed convex subset of a real Banach space $E$, $T_i : K \to K, (i = 1, 2, \ldots, p, p \geq 2)$ be $p$ uniformly $L_i$ Lipschitzian mappings with sequence $\{k_n\}_{n \geq 0} \subset [1, \infty), k_n \to 1$ and $\sum_{n \geq 0} (k_n - 1) < \infty$ such that $\rho \in \bigcap_{i=1}^{p \geq 2} F(T_i) \neq \phi$. Let $\{b_n\}_{n \geq 0}, \{b'_n\}_{n \geq 0}$ and $\{b''_n\}_{n \geq 0}$ be the real sequences in $[0,1]$ satisfying:

(i) $b_n, b'_n, b''_n \to 0$, as $n \to \infty, (i = 1, 2, \ldots, p - 2)$.

(ii) $\sum_{n \geq 0} b_n = \infty$.

For any $x_0 \in K$, define $\{x_n\}_{n \geq 0}$ by the iterative process (2.3). Suppose there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty), \Phi(0) = 0$ such that

$$< T_i^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|) \quad (2.4)$$

$\forall x \in K, (i = 1, 2, \ldots, p, p \geq 2)$. Then $\{x_n\}_{n \geq 0}$ converges strongly to $\rho \in \bigcap_{i=1}^{p \geq 2} F(T_i)$.

Proof: Since $T_1, T_2, \ldots, T_p, \ p \geq 2$ are uniformly $L_i$-Lipschitzian mappings, we have that $\forall x, y \in K$

$$\|T_i^n x - T_i^n y\| \leq L_i \|x - y\|, \ (i = 1, 2, \ldots, p).$$

Let $L = \max \{L_1, L_2, \ldots, L_p\}$.

Since $b_n, b'_n, b''_n \to 0$, as $n \to \infty, (i = 1, 2, \ldots, p - 2)$ there exists $n_0 \in N$ such that $\forall n \geq n_0$,

$$b_n \leq \min \left\{ \frac{1}{2 + 3L}, \frac{\Phi(2(\Phi^{-1}(a_0)))}{18(2 + 3L)\Phi(\Phi^{-1}(a_0))^2} \right\},$$
In addition, we obtain the following estimates

\[ b_{n-2}^{p-2} \leq \min \left\{ \frac{1}{2 + 3L}, \frac{\Phi(2(\Phi^{-1}(a_0)))}{18L(2 + 3L)\Phi^2(a_0)} \right\}, \quad (2.5) \]

\[ b^{p-1} \leq \frac{1}{2}\left\{ \frac{1}{1 + L} \right\} \]

and

\[ k_n - 1 \leq \frac{\Phi(2(\Phi^{-1}(a_0)))}{54\Phi^2(a_0)}. \quad (2.6) \]

Define \( a_{0,i} := \|x_{n_0} - T^1 x_{n_0}\|\|x_{n_0} - \rho\| + (k_n - 1)\|x_{n_0} - \rho\|^2 \) \((i = 1, 2, \ldots, p, p \geq 2)\) such that

\[ a_0 = \max \{x_{0,1}, x_{0,2}, \ldots, x_{0,p}\}. \]

Then by (2.4) we get

\[ \|x_n - \rho\| \leq \Phi^{-1}(a_0) \quad (2.7) \]

We claim that \( \|x_n - \rho\| \leq 2\Phi^{-1}(a_0) \) \( \forall n \geq n_0. \)

By (2.7), the claim holds for \( n=0. \) Assumed that \( \|x_n - \rho\| \leq 2\Phi^{-1}(a_0) \) holds for some \( n, \) we shall prove by contradiction that \( \|x_{n+1} - \rho\| \leq 2\Phi^{-1}(a_0). \) That is, \( \|x_{n+1} - \rho\| > 2\Phi^{-1}(a_0). \)

Using the iteration process (2.3) we have

\[ \|y_{n}^{p-1} - \rho\| = \|(1 - b_n^{p-1})x_n + b_n^{p-1}T_p^nx_n - \rho\| \]

\[ = \| (x_n - \rho) - b_n^{p-1}(x_n - T_p^nx_n)\| \]

\[ \leq \| x_n - \rho\| + b_n^{p-1}(\|x_n - \rho\| + \|T_p^nx_n - \rho\|) \]

\[ \leq 2\Phi^{-1}(a_0) + b_n^{p-1}(\|x_n - \rho\| + L\|x_n - \rho\|) \]

\[ \leq 2\Phi^{-1}(a_0) + 2(1 + L)\Phi^{-1}(a_0)b_n^{p-1} \]

\[ \leq 3\Phi^{-1}(a_0). \]

\[ \|y_{n}^{p-2} - \rho\| = \|(1 - b_n^{p-2})x_n + b_n^{p-2}T_{p-1}^ny_n^{p-1} - \rho\| \]

\[ = \| x_n - \rho\| + b_n^{p-2}(\|T_{p-1}^ny_n^{p-1} - \rho\|) \]

\[ \leq \| x_n - \rho\| + b_n^{p-2}(L\|y_n^{p-1} - \rho\| + \|x_n - \rho\|) \]

\[ \leq 2\Phi^{-1}(a_0) + (2 + 3L)\Phi^{-1}(a_0)b_n^{p-2} \]

\[ \leq 3\Phi^{-1}(a_0). \]

Recursively, we have \( \|y_i^{p-1} - \rho\| \leq 3\Phi^{-1}(a_0). \) Thus, \( \|y_i^{p} - \rho\| \leq 3\Phi^{-1}(a_0) \) for \((i = 1, 2, \ldots, p - 2).\)

In addition, we obtain the following estimates

\[ \| x_n - T^ny_n^i \| \leq \|x_n - \rho\| + \|T_i^ny_n^i - \rho\| \]

\[ \leq \|x_n - \rho\| + L\|y_n^i - \rho\| \]

\[ \leq (2 + 3L)\Phi^{-1}(a_0). \]
\[ \| x_{n+1} - \rho \| \leq 3 \Phi^{-1}(a_0). \]

\[ \| x_{n+1} - x_n \| \leq (2 + 3L)b_n \Phi^{-1}(a_0). \]

\[ \| x_n - T^ny_n^{-1} \| \leq (2 + 3L)b_n \Phi^{-1}(a_0). \]

\[ \| y_n^{p-2} - x_{n+1} \| \leq \| y_n^{p-2} - x_n \| + \| x_{n+1} - x_n \| \]

\[ \leq b_n^{p-2} \| T_{n-1}^ny_n^{-1} - x_n \| + \| x_{n+1} - x_n \| \]

\[ \leq (2 + 3L)b_n^{p-2}\Phi^{-1}(a_0) + (2 + 3L)b_n \Phi^{-1}(a_0). \]

Applying the estimates above and Lemma 2.10, we have

\[ \| x_{n+1} - \rho \|^2 = \| (1 - b_n)x_n + b_nT_1ny_n^1 - \rho \|^2 \]

\[ = \| x_n - \rho - b_n(T_1ny_n^1 - x_n) \|^2 \]

\[ \leq \| x_n - \rho \|^2 - 2b_n < T_1ny_n^1 - x_n, j(x_{n+1} - \rho) > \]

\[ = \| x_n - \rho \|^2 - 2b_n < T_1ny_n^1 - x_n, j(x_{n+1} - \rho) > \]

\[ - 2b_n < x_{n+1} - \rho, j(x_{n+1} - \rho) > \]

\[ + \| 2b_n < T_1ny_n^1 - T_1nx_{n+1}, j(x_{n+1} - \rho) > \]

\[ + 2b_n < x_{n+1} - x_n, j(x_{n+1} - \rho) > \]

\[ \leq \| x_n - \rho \|^2 + 2b_n(k_n + 1)\| x_n - \rho \|^2 - 2b_n\Phi(\| x_{n+1} - \rho \|) \]

\[ - 2b_n\| x_{n+1} - \rho \|^2 + 2b_n\| T_1ny_n^1 - T_1nx_{n+1} \| \| x_{n+1} - \rho \| \]

\[ + 2b_n\| x_{n+1} - x_n \| \| x_{n+1} - \rho \| \]

\[ = \| x_n - \rho \|^2 + 2b_n(k_n - 1)\| x_{n+1} - \rho \|^2 - 2b_n\Phi(x_{n+1} - \rho) \]

\[ + 2b_nL\| y_n^1 - x_{n+1} \| \| x_{n+1} - \rho \| + 2b_n\| x_{n+1} - x_n \| \| x_{n+1} - \rho \| \]

\[ \leq \| x_n - \rho \|^2 - 2b_n\Phi(2(\Phi^{-1}(a_0))) + 2b_n(k_n - 1)\| x_{n+1} - \rho \|^2 \]

\[ + 2b_nL[(2 + 3L)b_n^{p-2}\Phi^{-1}(a_0)]\| x_{n+1} - \rho \| \]

\[ + 2b_n(2 + 3L)b_n \Phi^{-1}(a_0))\| x_{n+1} - \rho \| \]

\[ \leq \| x_n - \rho \|^2 - 2b_n\Phi(2(\Phi^{-1}(a_0))) + 18b_n(k_n - 1)(\Phi^{-1}(a_0))^2 \]

\[ + 6b_nk_n^{p-2}L((2 + 3L)(\Phi^{-1}(a_0))^2 + 6b_n(2 + 3L)(\Phi^{-1}(a_0))^2 \]

\[ \leq \| x_n - \rho \|^2 - 2b_n\Phi(2(\Phi^{-1}(a_0))) + b_n\Phi(2(\Phi^{-1}(a_0))) \]

\[ = \| x_n - \rho \|^2 - b_n\Phi(2(\Phi^{-1}(a_0))) \]

\[ \leq (2(\Phi^{-1}(a_0)))^2. \]

This is a contradiction.

Hence, \( \| x_{n+1} - \rho \| \leq 2\Phi^{-1}(a_0) \). Therefore, \( \{x_n\} \) is bounded so also are
\( \{y_n\}, \{T_n^nx_n\}, \{T_{n-1}^ny_n^{-1}\} \) and \( \{T_i^ny_n^i\}, (i = 1, 2, \ldots, p - 2) \).

Now from \((**)*\), we have,
\[ \|x_{n+1} - \rho\|^2 \leq \|x_n - \rho\|^2 - 2b_n \Phi(2(\Phi^{-1}(a_o))) \\
+ 8b_n(k_n - 1)(\Phi^{-1}(a_o))^2 + 4b_nb_p^{p-2}L((2 + 3L)(\Phi^{-1}(a_o))^2 \\
+ 4b_n^2(2 + 3L)(\Phi^{-1}(a_o))^2 \\
= \|x_n - \rho\|^2 - 2b_n \Phi(2(\Phi^{-1}(a_o))) \\
+ 4[\Phi^{-1}(a_o))^2(2(k_n - 1) + (2 + 3L)[Lb_p^{p-2} + b_n]]b_n. \]

Taking \(d_n = \|x_n - \rho\|, e_n = 2b_n,\)
\(e(\tilde{e}_n) = 4[\Phi^{-1}(a_o))^2(2(k_n - 1) + (2 + 3L)[Lb_p^{p-2} + b_n]]b_n.\)

Then, using Lemma 1.2 and the conditions of the parameters, we obtain
\(d_n \to 0\) as \(n \to \infty\). This ends the proof.

**Corollary 3.2.** (i) Taking \(b_p^{p-1} = b_i^n = 0, i = 3, 4, ..., p - 2\) in the Theorem (3.1) yields Theorem (2.6) of Rafiq et al.[6]

(ii) Also, if \(b_p^{-1} = b_i^n = 0, i = 4, 5, ..., p - 2\) in Theorem 3.1 we obtained the result of Mogbademu [6, Theorem 2.1].

**Remark 3.3.** Theorem 3.1 in the sequel answered the question posed by Mogbademu [6].

**References**


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