On the Probability of Hitting a Constant or a Time-Dependent Boundary for a Geometric Brownian Motion with Time-Dependent Coefficients

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Abstract

This paper presents exact analytical formulae for the crossing of a constant one-sided or two-sided boundary by a geometric Brownian motion with time-dependent, non-random, drift and diffusion coefficients, under the assumption that the drift coefficient is a constant multiple of the diffusion coefficient, as well as approximate analytical formulae for general time-dependent, non-random, drift and diffusion coefficients and general time-dependent, non-random, boundaries. The numerical implementation of these formulae is very simple.

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1 Introduction

The question of the hitting time of a diffusion process to an absorbing boundary is of central importance in many applied mathematical sciences. It appears as a classical modelling framework in various branches of physics but it is also at the
core of more recent subjects such as quantitative finance. Typically, one needs to compute the probability that some random dynamics that can be modelled as a diffusion process will remain under or above some critical threshold over a given time interval. Almost all known exact results rely on the assumption that both the coefficients of the diffusion process and the boundary are constant, whereas in practice time-dependent diffusion process coefficients and boundaries are required for modeling purposes. As a result, slow and tedious numerical schemes need to be implemented, the accuracy of which may be dubious.

Only two cases of time-dependent boundaries are easily handled from a mathematical point of view: the crossing of a linear barrier by a Brownian motion with constant drift and diffusion coefficients (see Doob [7] and Sheike [24] for a slight generalization) and the crossing of the exponential of a linear barrier by a geometric Brownian motion with constant drift and diffusion coefficients (see Kunitomo & Ikeda [13]).

Some papers provide semi-analytical results for boundaries that are fairly general functions of time. Durbin [8], Ferebee [9] and Park & Schuurmann [19] thus derive integral equations, which must be numerically solved. Salminen [23] obtains a formula involving the expected value of a Brownian functional requiring to solve a non-time homogeneous Schrödinger equation with boundary and initial conditions. Nonlinear boundaries are studied in a general framework in Jennen & Lerche [11], Daniels [6], as well as Alili & Patie [1].

Other contributions deal with specific forms of the boundary. Breiman [3] thus studies the case of a square root boundary and relates it to the question of the first hitting time of a constant boundary by an OU (Ornstein-Uhlenbeck) process, while Groeneboom [10] examines the case of a quadratic function of time and shows that the first passage density can be written as a functional of a Bessel process of dimension 3.


Finally, a collection of papers focuses on applications to financial mathematics, more specifically to option pricing, which shifts the focus to geometric Brownian motion, as the latter serves as a the building block for modeling asset prices. Kunitomo & Ikeda [13] show that exponential boundaries are a simple extension to the standard two-sided barrier option pricing problem with constant boundaries. Roberts & Shortland [20] obtain tight bounds on the prices of barrier options with general moving boundaries and non constant coefficients by means of a hazard rate tangent approximation involving some numerical integration. Rogers & Zane [22] use a trinomial tree approach combined with a transformation of the statespace and a time change that turn smoothly-moving barrier option pricing problems into fixed barrier problems. Their result builds on a binomial lattice.
approach by Rogers & Stapleton [21]. Novikov et al. ([16], [17]) compute piecewise linear approximations for one-sided and two-sided boundary crossing probabilities using repeated numerical integration and apply this method to the pricing of time-dependent barrier options. Thompson [25] derives upper and lower bounds in the form of double integrals that prove to be slightly tighter but harder to compute than those of Roberts & Shortland [20]. Lo et al. [14] also provide tight bounds for barrier option prices by means of a multistage approximation scheme in terms of multivariate normal distribution functions which involve non-trivial numerical integration issues.

For all this rather abundant published research, there is no known formula for the crossing of a time-dependent absorbing boundary by a diffusion process with time-dependent drift and diffusion coefficients in general. The purpose of this article is two-fold:

- to provide exact analytical formulae for the crossing of a constant one-sided or two-sided boundary by a geometric Brownian motion with time-dependent, non-random, drift and diffusion coefficients, under the assumption that the drift coefficient is a constant multiple of the diffusion coefficient
- to provide approximate analytical formulae for general time-dependent, non-random, drift and diffusion coefficients and general time-dependent, non-random, boundaries

The method used is a combination of partial differential equations and changes of probability measure. All formulae are very easily implemented and yield instantaneous numerical values.

Section 2 provides exact analytical results under restrictive assumptions on the time-dependent diffusion process coefficients. Section 3 provides approximate analytical results for general time-dependent diffusion process coefficients and boundaries. Section 4 aims at testing the accuracy of the approximation formulae given in Section 3.

2 Exact solutions in particular cases

This section presents two exact closed form results that hold when the time-dependent drift coefficient is a constant multiple of the time-dependent diffusion coefficient and the boundary is constant. The first result is called Proposition 1. It provides the joint cumulative distribution function of not hitting a one-sided constant boundary during a finite time interval \([0, T]\) and of being under a given point at time \(T\). The second result is called Proposition 2. It provides the joint cumulative distribution function of not hitting a two-sided constant boundary during a finite time interval \([0, T]\) and of being above or under a given point at time \(T\).

**PROPOSITION 1**

Let \(B(t)\) be a standard Brownian motion with natural filtration \(\mathcal{F}_t\).
Let \( \sigma(t) \) be a piecewise continuous, non-random, positive real function such that:

\[
0 < \int_0^T \sigma^2(t) \, dt < \infty, \quad \forall \, T : 0 \leq T < \infty
\]  

Let \( m \) be any real constant. Let \( X(t) \) be a geometric Brownian motion such that \( X(0) = x_0 > 0 \). Under a given measure \( \mathbb{P} \), \( X(t) \) is driven by the following stochastic differential equation:

\[
dX(t) = m \sigma^2(t) X(t) \, dt + \sigma(t) X(t) \, dB(t)
\]  

Let \( h_0 \) be a positive real constant such that \( h_0 > x_0 \) and \( k \) be a positive real constant such that \( k < h_0 \). Let \( N(\cdot, \cdot) \) denote the cumulative distribution function of a standard normal random variable.

Then, the probability that \( X(t) \) will not hit \( h_0 \) during the finite time interval \([0, T]\) and that it will be below \( k \) at time \( T \) is given by:

\[
\mathbb{P} \left( \sup_{0 \leq t \leq T} X(t) < h_0, X(T) < k \right)
= \left( \frac{x_0}{h_0} \right)^{1-m} N \left( \frac{\ln \left( \frac{k}{x_0} \right) - \frac{1}{2} \int_0^T \sigma^2(t) \, dt}{\sqrt{\int_0^T \sigma^2(t) \, dt}} \right) - \left( \frac{h_0}{x_0} \right)^{1-m} N \left( \frac{\ln \left( \frac{k h_0^2}{h_0^2} \right) - \frac{1}{2} \int_0^T \sigma^2(t) \, dt}{\sqrt{\int_0^T \sigma^2(t) \, dt}} \right)
\]  

**Proof of Proposition 1**

As a consequence of the theory of absorbed diffusions in general and of the Fokker-Planck equation in particular (see, e.g., Oksendal [18]), the sought probability, denoted by \( p(x,t) \), is the solution of the following IBVP (initial boundary value problem) (4)-(5):

\[
\frac{\partial p}{\partial t} = m \sigma^2(t) x \frac{\partial p}{\partial x} + \sigma^2(t) x^2 \frac{\partial^2 p}{\partial x^2}, \quad 0 < t < T, \quad 0 < x < h_0, \quad h_0 > x_0
\]  

\[
\lim_{x \to 0} p(t,x) = 1, \quad p(t,h_0) = 0, \quad p(0,x) = U(k-x)
\]  

where \( U(\cdot) \) stands for the Heaviside function.

The following change of the space coordinate:

\[
y = \ln \left( \frac{x}{x_0} \right)
\]  

yields a new IBVP (7)-(8):

\[
\frac{\partial p}{\partial t} = \sigma^2(t) \left( m - \frac{1}{2} \frac{\partial p}{\partial y} \right) + \sigma^2(t) \frac{\partial^2 p}{2 \partial y^2}, \quad 0 < t < T, \quad -\infty < y < \ln \left( \frac{h_0}{x_0} \right)
\]  

\[
\lim_{y \to -\infty} p(t,y) = 1, \quad p\left(t, \ln \left( \frac{h_0}{x_0} \right) \right) = 0, \quad p(0,y) = U(k-x_0 \exp(y))
\]  

The following change of function:
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\[ p(y,t) = \exp\left(-\left(m - \frac{1}{2}\right)y - \frac{1}{8} \int_{0}^{t} \sigma^2(s) \, ds\right) w(y, t) \]  \hspace{1cm} (9)

along with the following change of variable:

\[ z = y - \ln\left(\frac{h_t}{x_0}\right) \]  \hspace{1cm} (10)

then yields the new IBVP:

\[ \frac{\partial w}{\partial t} = \frac{\sigma^2(t)}{2} \frac{\partial^2 w}{\partial z^2}, \quad 0 < t < T, \quad -\infty < z < 0 \]  \hspace{1cm} (11)

\[ w(t,0) = 0, \quad w(0,z) = U(k - h_0 \exp(z)) \sqrt{h_0 \exp\left(\frac{z}{2}\right)} \]  \hspace{1cm} (12)

By separation of variables, the following function verifies equation (11) and the boundary condition in (12):

\[ w(z,t) = \int_{0}^{\infty} A(\lambda) \exp\left(-\frac{\lambda^2 t}{2} \int_{0}^{t} \sigma^2(s) \, ds\right) \sin(\lambda z) \, d\lambda \]  \hspace{1cm} (13)

As a consequence of the initial condition in (12), the function

\[ f(z) \triangleq U(k - h_0 \exp(z)) \sqrt{h_0 \exp\left(\frac{z}{2}\right)} \]  \hspace{1cm} (14)

can be identified as the sine Fourier transform of \( A(\lambda) \). Then, applying Fubini’s theorem, classical trigonometric identities and the following Fourier cosine transform:

\[ \int_{0}^{\infty} \exp\left(-b z^2\right) \cos(a z) \, dz = \frac{1}{2} \sqrt{\pi} \exp\left(-\frac{a^2}{4b}\right), \quad \forall (a, b) \in \mathbb{R}^2_+ \]  \hspace{1cm} (15)

one can obtain:

\[ w(z,t) = \sqrt{\frac{h_0}{x_0}} \int_{-\infty}^{\infty} \frac{1}{2\pi \int_{0}^{t} \sigma^2(s) \, ds} \exp\left(\frac{v}{2} - \frac{(v - z)^2}{2 \int_{0}^{t} \sigma^2(s) \, ds}\right) \frac{v}{2} \int_{0}^{t} \sigma^2(s) \, ds \, dv \]  \hspace{1cm} (16)

\[ \sqrt{-\frac{h_0}{x_0}} \int_{-\infty}^{\infty} \frac{1}{2\pi \int_{0}^{t} \sigma^2(s) \, ds} \exp\left(\frac{v}{2} + \frac{(v + z)^2}{2 \int_{0}^{t} \sigma^2(s) \, ds}\right) \frac{v}{2} \int_{0}^{t} \sigma^2(s) \, ds \, dv \]

Straightforward calculations yield:

\[ w(z,t) = \sqrt{\frac{h_0}{x_0}} \exp\left(\frac{z}{2} + \frac{1}{8} \int_{0}^{t} \sigma^2(s) \, ds\right) N\left[\sqrt{\frac{\ln\left(\frac{k}{h_0}\right)}{h_0} - \int_{0}^{t} \sigma^2(s) \, ds}\right] \]  \hspace{1cm} (17)
It can be checked that the function \( w(z,t) \) given in (17) verifies (11) and (12). Reverting to the initial function and variables, Proposition 1 ensues.

\[ \Box \]

**PROPOSITION 2**

Let \( B(t), \sigma(t) \) and \( X(t) \) be defined as in Proposition 1 and, without loss of generality, let us set \( m = 1 \) in (3). Let \( d_0 \) and \( u_0 \) be two positive real constants such that \( d_0 < x_0 < u_0 \) and let \( k \) be a positive real constant such that \( k > d_0 \).

Then, the probability that \( X(t) \) will hit neither \( d_0 \) nor \( u_0 \) during the finite time interval \([0,T]\) and that it will be above \( k \) at time \( T \) is given by:

\[
\mathbb{P}\left( \inf_{0 \leq t \leq T} X(t) > d_0, \sup_{0 \leq t \leq T} X(t) < u_0, X(T) > k \right)
= \frac{2}{\ln(x_0/d_0)} \exp\left( -\frac{1}{8} \int_0^t \sigma^2(s) \, ds \right) \sum_{n=1}^{\infty} \left( \frac{\ln(u_0/d_0)}{\ln(k/d_0)} \right)^n \exp\left( \frac{w}{2} \right) \sin\left( \frac{\pi k}{\ln(u_0/d_0)} \right) \; dw
\]

\[
\exp\left( -\frac{1}{2} \left( \frac{\pi k}{\ln(u_0/d_0)} \right)^2 \right) \int_0^t \sigma^2(s) \, ds \left( \frac{\pi k}{\ln(u_0/d_0)} \right) \sin\left( \frac{\pi k}{\ln(u_0/d_0)} \right) \right)\right)
\]

**Corollary**:

\[
\mathbb{P}\left( \inf_{0 \leq t \leq T} X(t) > d_0, \sup_{0 \leq t \leq T} X(t) < u_0, X(T) < k \right)
= \frac{2}{\ln(x_0/d_0)} \exp\left( -\frac{1}{8} \int_0^t \sigma^2(s) \, ds \right) \sum_{n=1}^{\infty} \left( \frac{\ln(u_0/d_0)}{\ln(k/d_0)} \right)^n \exp\left( \frac{w}{2} \right) \sin\left( \frac{\pi k}{\ln(u_0/d_0)} \right) \; dw
\]

\[
\exp\left( \int_0^t \sigma^2(s) \, ds \left( -\frac{1}{2} \left( \frac{\pi k}{\ln(u_0/d_0)} \right)^2 - \frac{1}{8} \right) \right) \sin\left( \frac{\pi k}{\ln(u_0/d_0)} \right) \left( \ln\left( \frac{x_0}{u_0} \right) \right)
\]

**Proof of Proposition 2**

Following the same steps as in the proof of Proposition 1, one can show that the probability under consideration is the function \( w(z,t) \) solving the following IBVP (20)-(21):

\[
\frac{\partial w}{\partial t} = \frac{\sigma^2(t)}{2} \frac{\partial^2 w}{\partial z^2}, \quad 0 < t < T, \quad 0 < z < \ln\left( \frac{u_0}{d_0} \right)
\]

(20)
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\[ w(t,0) = 0, \quad w(t, \ln \left( \frac{u_0}{d_0} \right)) = 0, \quad w(0, z) = U \left( d_0 \exp(z) - k \right) \frac{d_0}{x_0} \exp \left( \frac{z}{2} \right) \]  

(21)

where:

\[ z = y - \ln \left( \frac{d_0}{x_0} \right), \quad y = \ln \left( \frac{x}{x_0} \right), \quad p(y, t) = \exp \left( \frac{-y}{2} - \frac{1}{8} \int_0^t \sigma^2(s) \, ds \right) w(y, t) \]  

(22)

Using separation of variables, the boundary and initial conditions in (21) and the superposition principle, one easily obtains:

\[ w(z, t) = \sum_{n=1}^{\infty} A_n \exp \left( -\frac{1}{2} \left( \frac{n\pi}{\ln(u_0/d_0)} \right)^2 \int_0^t \sigma^2(s) \, ds \right) \sin \left( \frac{n\pi}{\ln(u_0/d_0)} z \right) \]  

(23)

where:

\[ A_n = 2 \frac{\sqrt{d_0}}{x_0} \frac{\ln(u_0/d_0)}{\ln(k/d_0)} \int \exp \left( \frac{w}{2} \right) \sin \left( \frac{n\pi}{\ln(u_0/d_0)} w \right) \, dw \]  

(24)

Then, reverting to the initial variables and function, and substituting \( x = x_0 \), Proposition 2 ensues.

\[ \square \]

3 Approximate analytical solutions for general time-dependent, non-random boundary, drift and diffusion coefficients

This section provides approximate closed form analytical results for general non-random, time-dependent boundaries and process coefficients. The first result is called Proposition 3. It provides an approximate formula for the cumulative distribution function of not hitting a one-sided time-dependent, non-random boundary during a finite time interval \([0, T]\). The second result is called Proposition 4. It provides an approximate formula for the joint cumulative distribution function of not hitting a two-sided constant boundary during a finite time interval \([0, T]\) and of being below a given point at time \( T \). The final result is called Proposition 5 and provides the joint cumulative distribution function of not hitting a one-sided constant boundary during a finite time interval \([0, T]\) and of being under or above a given point at time \( T \).

PROPOSITION 3

Let \( B(t) \) and \( \sigma(t) \) be defined as in Proposition 1. Let \( \mu(t) \) be a piecewise continuous non-random real function such that:

\[ \int_0^T \frac{\mu^2(t)}{\sigma^2(t)} \, dt < \infty, \quad \forall T : 0 \leq T < \infty \]  

(25)

Under a given measure \( \mathbb{P} \), the process \( X(t) \) is driven by the following stochastic differential equation:
\[ dX(t) = \mu(t) X(t) \, dt + \sigma(t) X(t) \, dB(t) \quad (26) \]

where the usual growth conditions apply to \( \mu(t) \) and \( \sigma(t) \) for equation (26) to be properly defined (Oksendal [18]).

Let \( h(t) \) be a continuous, once differentiable real function, such that \( h(0) > x_0 \) and such that if there is a coefficient multiplying \( t \), then this coefficient is negative.

Under these assumptions, the probability that \( X(t) \) will not hit \( h(t) \) during the finite time interval \([0, T]\) can be approximated by the following formula:

\[
\mathbb{P}\left( X(t) < h(t), \forall t \leq T \right) \\
\approx N \left\{ \ln \left( \frac{h(0)}{x_0} \right) \right\} - \sqrt{\int_0^T \frac{\mu(t) - \frac{\sigma^2(t)}{2} - \frac{h'(t)}{h(t)}}{\sigma^2(t)} \, dt} \\
- \exp \left\{ \sqrt{\int_0^T \frac{\sigma^2(t)}{2} \, dt} \ln \left( \frac{h(0)}{x_0} \right) \right\} \\
\times N \left\{ \ln \left( \frac{x_0}{h(0)} \right) \right\} - \sqrt{\int_0^T \frac{\mu(t) - \frac{\sigma^2(t)}{2} - \frac{h'(t)}{h(t)}}{\sigma^2(t)} \, dt} 
\] \quad (27)

Remark: If we denote by \( \tau_{h(t)} \) the first passage time of the process \( X(t) \) to the time-dependent boundary \( h(t) \) in \([0, T]\), that is:

\[
\tau_{h(t)} = \inf \{ t \in [0, T] : X(t) = h(t) \} \quad (28)
\]

then the density of \( \tau_{h(t)} \) can be approximated by

\[
\mathbb{P}\left( \tau_{h(t)} \in dt \right) \approx \frac{\partial}{\partial t} \left[ \mathbb{P}\left( X(s) < h(s), \forall s \leq t \right) \right] \\
\] \quad (29)

where an approximation of \( \mathbb{P}\left( X(s) < h(s), \forall s \leq t \right) \) is given by (27).

**Proof of Proposition 3:**

As a consequence of the theory of absorbed diffusions in general and of the Fokker-Planck equation in particular (see, e.g., Oksendal [18]), the probability
under consideration, denoted by \( p(x,t) \), is the solution of the following IBVP (30)-(31):

\[
\frac{\partial p}{\partial t} = \mu(t)x \frac{\partial p}{\partial x} + \frac{\sigma^2(t)}{2} x^2 \frac{\partial^2 p}{\partial x^2}, \quad 0 < t < T, \quad 0 < x < h(t), \quad h(0) > x_0
\]

\[
\lim_{x \to 0} p(t,x) = 1, \quad \left. p(t,x) \right|_{x=h(t)} = 0, \quad p(0,x) = 1
\] (31)

The next transformations of the space coordinate:

\[
y = \ln \left( \frac{x}{x_0} \right), \quad z = y - \ln \left( \frac{h(t)}{x_0} \right)
\]

yield a new IBVP (33)-(34):

\[
\frac{\partial p}{\partial t} = \left( \mu(t) - \frac{\sigma^2(t)}{2} - \frac{h'(t)}{h(t)} \right) \frac{\partial p}{\partial z} + \frac{\sigma^2(t)}{2} \frac{\partial^2 p}{\partial z^2}, \quad 0 < t < T, \quad -\infty < z < 0
\]

\[
\lim_{z \to -\infty} p(t,z) = 1, \quad p(t,0) = 0, \quad p(0,z) = 1
\] (34)

If the function \( p(z,t) \) solving (33)-(34) can be obtained, then the sought probability will be equal to \( p \left( z = \ln \left( \frac{h(0)}{x_0} \right), t = T \right) \). Unfortunately, the IBVP (33)-(34) cannot be solved exactly by known methods. A probabilistic approach can nevertheless be applied to get an analytical approximation of the function \( p(z,t) \).

Set:

\[
\lambda(t) \triangleq \mu(t) - \frac{\sigma^2(t)}{2} - \frac{h'(t)}{h(t)}
\] (35)

Let a new process \( Y(t) \) be started from the origin, whose motion is driven by:

\[
dY(t) = \lambda(t) dt + \sigma(t) dB(t)
\] (36)

Then, solving IBVP (33)-(34) is the same as calculating the probability, under the measure \( \mathbb{P} \), that \( Y(t) \) will remain below the real number \( y_0 \triangleq \ln \left( \frac{h(0)}{x_0} \right) \) during the finite time interval \([0,T] \).

Let \( \mathbb{Q} \) be the measure under which the stochastic differential of \( Y(t) \) is given by:

\[
dY(t) = \sigma(t) dW(t)
\] (37)

where \( W(t) \) is a standard Brownian motion.

Then, using classical results on the hitting times of a standard Brownian motion (see, e.g., Karatzas & Shreve [12]), a simple time change of \( W(t) \) yields:

\[
\mathbb{Q} \left\{ \sup_{0 \leq t \leq T} Y(t) \leq y_0 \right\} = N \left( -\frac{y_0}{\sqrt{\int_0^T \sigma^2(s) \, ds}} \right) - N \left( -\frac{-y_0}{\sqrt{\int_0^T \sigma^2(s) \, ds}} \right)
\] (38)

The \( \mathbb{P} \)-measure is equivalent to \( \mathbb{Q} \) and its Radon-Nikodym density is given by:

\[
\frac{d\mathbb{P}}{d\mathbb{Q}} = \exp \left\{ \int_0^T \frac{\lambda(s)}{\sigma(s)} \, dW(s) - \frac{1}{2} \int_0^T \left( \frac{\lambda(s)}{\sigma(s)} \right)^2 \, ds \right\} \triangleq L_t
\] (39)

Existence of the integrals in (39) is guaranteed by the assumptions made on the functions...
Girsanov’s theorem (see, e.g., Karatzas & Shreve [12]) allows us to state that the process $B(t)$ defined by:

$$B(t) = W(t) - \int_0^t \frac{\lambda(s)}{\sigma(s)} \, ds$$  \hfill (40)

is a $\mathbb{P}$-Brownian motion.

The function $\frac{\lambda(t)}{\sigma(t)}$ is non-random. Hence, if $t$ is fixed:

$$\int_0^t \frac{\lambda(s)}{\sigma(s)} \, dW(s) \sim \mathcal{N} \left(0, \int_0^t \left(\frac{\lambda(s)}{\sigma(s)}\right)^2 \, ds\right)$$  \hfill (41)

where $\mathcal{N}(a,b)$ refers to the normal distribution with expectation $a$ and variance $b$.

Thus, if $t$ is fixed, the following equalities hold in law:

$$\int_0^t \frac{\lambda(s)}{\sigma(s)} \, dW(s) \overset{\text{law}}{=} \sqrt{t} \left(\int_0^t \left(\frac{\lambda(s)}{\sigma(s)}\right)^2 \, ds\right)^{1/2}$$  \hfill (42)

where $\phi$ is a standard normal random variable.

Using (38), (39) and (42), the sought probability $\mathbb{P}\left(\sup_{0 \leq t \leq T} Y(t) \leq y_0\right)$ can now be approximated as follows:

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} Y(t) \leq y_0\right) = \mathbb{E}_Q \left( L_t \mathbb{1}_{\left(\sup_{0 \leq t \leq T} Y(t) \leq y_0\right)} \right)$$  \hfill (43)

$$\approx \int_{-\infty}^{\gamma_0} \exp \left\{ t \sqrt{\int_0^t \left(\frac{\lambda(s)}{\sigma(s)}\right)^2 \, ds} - \frac{1}{2} \int_0^t \left(\frac{\lambda(s)}{\sigma(s)}\right)^2 \, ds \right\} \, dx$$  \hfill (44)

Straightforward computations yield:

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} Y(t) \leq y_0\right) \approx N \left(\frac{y_0}{J(t)} - J(t)\right) - \exp \left(2y_0 \frac{J(t)}{I(t)}\right) N \left(-\frac{y_0}{I(t)} - J(t)\right)$$  \hfill (45)

with:

$$I(t) = \sqrt{\int_0^t \sigma^2(s) \, ds}, \quad J(t) = \sqrt{\int_0^t \frac{\lambda^2(s)}{\sigma^2(s)} \, ds}$$  \hfill (46)
The result stated in (45) is just an approximation since (42) only holds for fixed $t$ and not as a stochastic differential, as shown by the fact that:

$$d \left( \frac{W(t)}{\sqrt{t}} \right) = \frac{\lambda(t)}{\sigma(t)} dW(t)$$

(47)

It can be checked that the function $p(z,t)$ defined by:

$$p(z,t) = N \left( \frac{z}{I(t)} - J(t) \right) - \exp \left( 2z \frac{J(t)}{I(t)} \right) N \left( - \frac{z}{I(t)} - J(t) \right)$$

(48)

satisfies the required boundary and initial conditions in (34) as well as the following PDE (partial differential equation):

$$\frac{\partial p}{\partial t} = \left( \mu(t) - \frac{\sigma^2(t)}{2} - \frac{h'(t)}{h(t)} \right) \frac{\partial p}{\partial z} + \frac{\sigma^2(t)}{2} \frac{\partial^2 p}{\partial z^2} + \Phi(z,t)$$

$$0 < t < T, \quad -\infty < z < 0$$

(49)

The function $\Phi(z,t)$ is a non-zero term that can be explicitly computed by substituting the function $p(z,t)$ given by (48) into the PDE given by (33). If the solution (48) were exact, then $\Phi(z,t)$ would be null. The residue $\Phi(z,t)$ becomes more and more negligible as the parameter $T$ in Proposition 3 decreases, so that the proposed analytical approximation should be very accurate on relatively “short” time intervals, as will be checked in Section 3.

Next, an approximation formula is provided when the boundary is constant and two-sided.

**PROPOSITION 4**

Let $B(t)$, $\mu(t)$, $\sigma(t)$ and $X(t)$ be defined as in Proposition 3. Let $d_0$ and $u_0$ be two positive real constants such that $d_0 < x_0 < u_0$ and let $k$ be a positive real constant such that $k > d_0$. Then, the probability that $X(t)$ will hit neither $d_0$ nor $u_0$ during the finite time interval $[0,T]$ and that it will be under $k$ at time $T$ can be approximated by:

$$\mathbb{P} \left( \inf_{0 \leq t \leq T} X(t) > d_0, \sup_{0 \leq t \leq T} X(t) < u_0, X(T) < k \right)$$
\[ z = \sum_{n=-\infty}^{\infty} \exp \left( \frac{2I_2(t)n\left( \ln \left( \frac{u_0}{x_0} \right) - \ln \left( \frac{d_0}{x_0} \right) \right)}{I_1(t)} \right) \left( \begin{array}{c} \frac{\ln \left( \frac{k_0}{x_0} \right) - 2n\left( \ln \left( \frac{u_0}{x_0} \right) - \ln \left( \frac{d_0}{x_0} \right) \right)}{I_1(t)} - I_2(t) \\ -N \left( \left( \ln \left( \frac{d_0}{x_0} \right) - 2n\left( \ln \left( \frac{u_0}{x_0} \right) - \ln \left( \frac{d_0}{x_0} \right) \right) \right) \right) - I_2(t) \end{array} \right) \right) \]

\[ - \sum_{n=-\infty}^{\infty} \exp \left( \frac{2I_2(t)n\left( \ln \left( \frac{d_0}{x_0} \right) - n\left( \ln \left( \frac{u_0}{x_0} \right) - \ln \left( \frac{d_0}{x_0} \right) \right)}{I_1(t)} \right) \right) \left( \begin{array}{c} N \left( \frac{\ln \left( \frac{k_0}{x_0} \right) - 2\ln \left( \frac{d_0}{x_0} \right) + 2n\left( \ln \left( \frac{u_0}{x_0} \right) - \ln \left( \frac{d_0}{x_0} \right) \right)}{I_1(t)} - I_2(t) \\ -N \left( \left( \ln \left( \frac{d_0}{x_0} \right) + 2n\left( \ln \left( \frac{u_0}{x_0} \right) - \ln \left( \frac{d_0}{x_0} \right) \right) \right) \right) - I_2(t) \end{array} \right) \right) \]

where:

\[ I(t) = \sqrt{\int_0^T \sigma^2(t) dt} \quad (51) \]

\[ I_2(t) = \sqrt{\int_0^T \left( \frac{\mu(t) - \sigma^2(t)}{2} \right)^2 dt} \quad (52) \]

**Proof of Proposition 4:**

Following the same steps as in the proof of Proposition 3, the probability under consideration, denoted by \( p(y,t) \), is the solution of the following IBVP (53)-(54):

\[ \frac{\partial p}{\partial t} = \left( \mu(t) - \frac{\sigma^2(t)}{2} \right) \frac{\partial p}{\partial y} + \frac{\sigma^2(t)}{2} \frac{\partial^2 p}{\partial y^2}, \quad 0 < t < T, \quad \ln \left( \frac{d_0}{x_0} \right) < y < \ln \left( \frac{u_0}{x_0} \right) \quad (53) \]

\[ p\left(t, \ln \left( \frac{d_0}{x_0} \right) \right) = 0, \quad p\left(t, \ln \left( \frac{u_0}{x_0} \right) \right) = 0, \quad p(0,y) = U(k - x_0 \exp(y)) \quad (54) \]

where \( y = \ln \left( \frac{x}{x_0} \right) \)

Set:
Geometric Brownian motion with time-dependent coefficients

$$\beta(t) \triangleq \mu(t) - \frac{\sigma^2(t)}{2}$$

(55)

Let a new process $Z(t)$ be started from the origin, whose motion is driven by:

$$dZ(t) = \beta(t) dt + \sigma(t) dB(t)$$

(56)

Solving IBVP (53)-(54) is the same as calculating the probability, under the measure $\mathbb{P}$, that $Z(t)$ will remain below the real number $\ln(\frac{u_0}{x_0})$ and above the real number $\ln(\frac{d_0}{x_0})$ during the finite time interval $[0,T]$ and that $Z(t)$ will be smaller than the real number $\ln(\frac{k}{x_0})$ at time $T$.

Let $\mathbb{Q}$ be the measure under which the stochastic differential of $Z(t)$ is given by:

$$dZ(t) = \sigma(t) dW(t)$$

(57)

where $W(t)$ is a standard Brownian motion.

Using the classical method of images (Carslaw & Jaeger [4]), one can obtain:

$$\mathbb{Q}\left( \inf_{0 \leq t \leq T} Z(t) \geq \ln(\frac{d_0}{x_0}), \sup_{0 \leq t \leq T} Z(t) \leq \ln(\frac{u_0}{x_0}), Z(T) < \ln(\frac{k}{x_0}) \right)$$

$$= \sum_{n=-\infty}^{\infty} \left[ N \left( \ln(\frac{k}{x_0}) - 2n(\ln(\frac{u_0}{x_0}) - \ln(\frac{d_0}{x_0})) \right) \right]$$

$$- \sum_{n=-\infty}^{\infty} \left[ N \left( \ln(\frac{d_0}{x_0}) - 2n(\ln(\frac{u_0}{x_0}) - \ln(\frac{d_0}{x_0})) \right) \right]$$

$$= \sum_{n=-\infty}^{\infty} \left[ N \left( \ln(\frac{k}{x_0}) - 2n(\ln(\frac{u_0}{x_0}) + 2n(\ln(\frac{u_0}{x_0}) - \ln(\frac{d_0}{x_0})) \right) \right]$$

$$- \sum_{n=-\infty}^{\infty} \left[ N \left( -\ln(\frac{d_0}{x_0}) + 2n(\ln(\frac{u_0}{x_0}) - \ln(\frac{d_0}{x_0})) \right) \right]$$

$$\mathbb{P} - \text{measure is equivalent to } \mathbb{Q} \text{ and its Radon-Nikodym density is given by :}$$

$$\frac{d\mathbb{P}}{d\mathbb{Q}} \bigg|_{\mathcal{F}_t} = \exp \left( \int_0^t \frac{\beta(s)}{\sigma(s)} dW(s) - \frac{1}{2} \int_0^t \left( \frac{\beta(s)}{\sigma(s)} \right)^2 ds \right) \triangleq L_t$$

(59)
Along the lines of the proof of Proposition 3, it can then be shown that the sought probability can be approximated by:

\[
\mathbb{P}\left( \inf_{0 \leq t \leq T} Z(t) \geq \ln\left( \frac{d_0}{x_0} \right), \sup_{0 \leq t \leq T} Z(t) \leq \ln\left( \frac{u_0}{x_0} \right), Z(T) < \ln\left( \frac{k}{x_0} \right) \right) = \mathbb{E}_{Q}\left( L_{t} \mathbb{P}_{t} \left( \inf_{0 \leq t \leq T} Z(t) \geq \ln\left( \frac{d_0}{x_0} \right), \sup_{0 \leq t \leq T} Z(t) \leq \ln\left( \frac{u_0}{x_0} \right), Z(T) < \ln\left( \frac{k}{x_0} \right) \right) \right) \tag{60}
\]

where the expectation operator in (60) is expanded into integral form using (58) and the following equality in law, for fixed \( t \):

\[
\int_{0}^{t} \frac{\beta(s)}{\sigma(s)} dW(s) \overset{\text{law}}{=} \phi \sqrt{\int_{0}^{t} \left( \frac{\beta(s)}{\sigma(s)} \right)^2 ds}, \phi \sim \mathcal{N}(0,1) \tag{61}
\]

Performing the necessary calculations, the formula in (50) is then obtained.

Eventually, an approximation formula is stated when the boundary is constant and one-sided.

**PROPOSITION 5**

Let \( B(t), \mu(t), \sigma(t) \) and \( X(t) \) be defined as in Proposition 3. Let \( h_0 \) be a positive real constant such that \( h_0 > x_0 \) and \( k \) be a positive real constant such that \( k < h_0 \). Then, the probability that \( X(t) \) will not hit \( h_0 \) during the finite time interval \([0, T]\) and that it will be below \( k \) at time \( T \) can be approximated by:

\[
\mathbb{P}\left( \sup_{0 \leq t \leq T} X(t) < h_0, X(T) < k \right)
\approx N \left( \frac{\ln\left( \frac{k}{x_0} \right)}{\sqrt{\int_{0}^{T} \sigma^2(t) dt}} - \frac{T}{\sigma^2(t)} \int_{0}^{T} \left( \frac{\mu(t) - \sigma^2(t)}{2} \right)^2 dt \right)
\]

\[
- \exp \left[ -\sqrt{2} \left( \frac{T}{\sigma^2(t)} \int_{0}^{T} \left( \frac{\mu(t) - \sigma^2(t)}{2} \right)^2 dt \right) \ln\left( \frac{h_0}{x_0} \right) \right]
\left( \frac{\ln\left( \frac{kx_0}{h_0^2} \right)}{\sqrt{\int_{0}^{T} \sigma^2(t) dt}} - \frac{T}{\sigma^2(t)} \int_{0}^{T} \left( \frac{\mu(t) - \sigma^2(t)}{2} \right)^2 dt \right)
\]  

Corollary: Let \( h_0 \) be a positive real constant such that \( h_0 > x_0 \) and \( k \) be a positive real constant such that \( k > h_0 \). Then, the probability that \( X(t) \) will not hit \( h_0 \) during the finite time interval \([0, T]\) and that it will be above \( k \) at time \( T \) is given by:
The proof is similar to that of Proposition 3, so the details are omitted.

4 Numerical results

In this final section, the accuracy of the analytical approximations provided by Proposition 3, Proposition 4 and Proposition 5, is numerically tested, by comparing obtained values with selected benchmarks.

The first stage of the procedure is to randomly draw parameters for the diffusion process \( X(t) \) defined in Proposition 3. Three different functional forms of coefficients \( \mu(t) \) and \( \sigma(t) \) are selected. The linear form reads:

\[
\mu(t) = a_1 + b_1 t, \quad \sigma(t) = a_2 + b_2 t
\]

The quadratic form reads:

\[
\mu(t) = a_1 + b_1 t + c_1 t^2, \quad \sigma(t) = a_2 + b_2 t + c_2 t^2
\]

The square root form reads:

\[
\mu(t) = \sqrt{a_1 + b_1 t}, \quad \sigma(t) = \sqrt{a_2 + b_2 t}
\]

The real constants \( a_1, a_2, b_1, b_2, c_1, c_2 \) are randomly drawn within wide predefined ranges. For instance, if the linear form is prescribed for the volatility function \( \sigma(t) \), the initial volatility \( \sigma(0) = a_2 \) may vary from 5% to 25%, while the coefficient \( b_2 \), measuring the increase in volatility over time, may vary from 30% to 80%. Depending on the random value of parameter \( T \), final values of the volatility function, \( \sigma(T) \), of up to 85% were observed during the numerical experiment.

The initial value of the process, \( X(0) \), is set at 1. To test Proposition 4 and Proposition 5, the parameters \( u_0, d_0, h_0 \) and \( k \) are randomly drawn. The predefined ranges for \( u_0, h_0 \) and \( d_0 \) are \( X(0) \pm [10\%;40\%] \). The parameter \( k \)
may vary between $X(0) - 10\%$ and $X(0) + 10\%$. To test Proposition 3, a time-dependent boundary $h(t)$ is also randomly picked that can take on the functional forms (64)-(65). Finally, the width of the time interval may vary from 0 to 1.

The next stage in the procedure is to define a benchmark that is assumed to be the “true” value. Possible benchmarks are the numerical solutions of the initial boundary value problems obtained in (33)-(34) and (53-54). This approach was attempted by means of classical Crank-Nicolson finite differences and led to quite unreliable results, even with fine meshes. Alternatively, one can resort to the Monte Carlo simulation of diffusion process $X(t)$ and of time dependent boundary $h(t)$ as far as Proposition 3 is concerned. This approach is both simple and robust. It was implemented using a standard Euler scheme for the discretization of the stochastic differential equation (26) with a timestep equal to $1/16000$. A rather fine timestep such as the latter is needed, otherwise too many boundary crossings would be missed. Still, plain Monte Carlo simulation is notoriously inaccurate. Besides increasing the number of simulations and using a reliable random number generator, it is advisable to have exact analytical benchmarks at disposal, with which one can test the accuracy of the Monte Carlo simulation. This can be achieved by means of Proposition 1 and Proposition 2 for the special case in which $\mu(t) = \sigma^2(t)$. So, our numerical procedure is performed twice for each set of process coefficients and boundaries: (i) drawing a general $\mu(t)$ in the first place to test the accuracy of Proposition 3, Proposition 4 and Proposition 5, in comparison with a Monte Carlo approximation used as a benchmark; (ii) setting $\mu(t) = \sigma^2(t)$ and using Proposition 1 and Proposition 2 as exact benchmarks to test the accuracy of the Monte Carlo approximation itself.

The second stage of the procedure is carried out using the same random numbers that are generated to compute a Monte Carlo approximation for a general $\mu(t)$ in the first place.

Tables 1 - 4 report the obtained results when conducting the numerical experiment above described. A sample of 100,000 collections of randomly drawn parameters is used in each Table. Three main indicators are reported:
- the observed mean absolute value of the error
- the observed highest absolute value of the error
- the observed proportions of errors that are: less than $\pm0.5\%$, between 0.5% and 1%, between 1% and 2%, between 2% and 3%, and over 3%

In Table 1, the “error” under consideration is the difference between the analytical approximations provided by Proposition 4, Proposition 5 and Proposition 3 on the one hand, and Monte Carlo approximations used as benchmarks on the other hand. The error is expressed as a percentage of the numerical value of the Monte Carlo approximation.

In Table 2, the “error” under consideration is the difference between a Monte Carlo approximation and exact analytical benchmarks provided by Proposition 1 and Proposition 2.

Table 1 shows that the mean absolute value of the error entailed by the use of Proposition 4, Proposition 5 and Proposition 3, is always less than 1%. But, in
practice, it is obviously important to have an idea of the order of magnitude of the greatest possible error incurred by a single use of an analytical approximation. Over the fairly large and uniformly distributed sample of random parameters that was used, it turns out that the maximum absolute value of the error is always less than 3% for Proposition 4 and 5, and always less than 5% for Proposition 5. These results are all the more reassuring as one must bear in mind that the Monte Carlo approximations used as benchmarks in Table 1 are relatively inaccurate, despite the high number of simulations and the fine timestep. Indeed, Table 2 shows that the absolute error entailed by the use of a Monte Carlo approximation as a benchmark may be as high as 1.67%. This suggests that the maximum absolute value of error entailed by Proposition 4, Proposition 5 and Proposition 3 may actually be smaller than the figures reported in Table 1.

As pointed out in Section 3, the magnitude of the errors entailed by the analytical approximations is a function of the width of the time interval $[0,T]$. So, in Tables 3 and 4, three ranges for $T$ are examined: $T < 0.1$, $T \in [0.1,0.25]$, and $T \in [0.25,1]$. Only Proposition 4 and Proposition 3 are tested, as the results obtained for Proposition 5 are very similar to those obtained for Proposition 4. It turns out that, when $T < 0.1$, which includes time intervals that may contain as many as 1,600 time steps, the maximum absolute value of the error entailed by Proposition 4 is only 0.58%, while 93% of observed errors are less than 0.5%. The figures are roughly the same for Proposition 3. These numerical results suggest that our analytical approximations should be quite relevant for practical purposes in a variety of applied sciences. Indeed, not only do these analytical approximations provide a framework in which the impact and the interactions of variables can be mathematically analyzed, but they are also extremely efficient compared with the slowness of a Monte Carlo simulation. To get a single Monte Carlo estimate in Table 1, one must draw between 80,000,000 and 8,000,000,000 uniform random numbers as the parameter $T$ varies between 0.01 and 1, which may typically take up to half an hour on a single PC. Furthermore, the issue of the reliability of the random number generator becomes more acute as the number of draws increases. In contrast, it always takes less than one second to obtain an estimate using Proposition 3, Proposition 4 or Proposition 5, as the functions involved are standard and the number of function evaluations implied by the numerical integrations is very modest.

Eventually, one can also wonder whether the magnitude of the errors entailed by the analytical approximations may vary according to the functional forms of the diffusion coefficients $\mu(t)$ and $\sigma(t)$, as well as to the functional form of the time dependent boundary $h(t)$. Numerical tests were implemented in this regard and showed no significance of that factor, as the orders of magnitude remained the same for all different functional forms of the diffusion process coefficients and of the time dependent boundary.
Table 1 – Numerical assessment of the error entailed by the use of the analytical approximations provided by Proposition 4, Proposition 5 and Proposition 3

<table>
<thead>
<tr>
<th></th>
<th>Proposition 4</th>
<th>Proposition 5</th>
<th>Proposition 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean absolute value of error</td>
<td>0.85%</td>
<td>0.77%</td>
<td>0.81%</td>
</tr>
<tr>
<td>maximum absolute value of error</td>
<td>2.76%</td>
<td>2.62%</td>
<td>4.87%</td>
</tr>
<tr>
<td>Proportion of $</td>
<td>\text{error}</td>
<td>&lt; 0.5%$</td>
<td>39%</td>
</tr>
<tr>
<td>Proportion of $</td>
<td>\text{error}</td>
<td>\in [0.5%;1%]$</td>
<td>37%</td>
</tr>
<tr>
<td>Proportion of $</td>
<td>\text{error}</td>
<td>\in [1%;2%]$</td>
<td>15%</td>
</tr>
<tr>
<td>Proportion of $</td>
<td>\text{error}</td>
<td>\in [2%;3%]$</td>
<td>9%</td>
</tr>
<tr>
<td>Proportion of $</td>
<td>\text{error}</td>
<td>&gt; 3%$</td>
<td>0%</td>
</tr>
</tbody>
</table>

Notes: The error is measured as the divergence from a Monte Carlo approximation used as a benchmark, and expressed as a percentage of the value of the Monte Carlo approximation. A sample of 100,000 different sets of parameters for the process $X(t)$ is considered. Details about the way the parameters are drawn can be found at the beginning of Section 4. For each set of parameters, a total of 500,000 Monte Carlo simulations are carried out to obtain a Monte Carlo benchmark, using a timestep equal to $1/16,000$ for the discretization scheme of the stochastic differential of $X(t)$.

Table 2 – Numerical assessment of the error entailed by the use of a Monte Carlo approximation as a benchmark

<table>
<thead>
<tr>
<th></th>
<th>Proposition 1</th>
<th>Proposition 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean absolute value of error</td>
<td>0.54%</td>
<td>0.58%</td>
</tr>
<tr>
<td>maximum absolute value of error</td>
<td>1.59%</td>
<td>1.67%</td>
</tr>
<tr>
<td>Proportion of $</td>
<td>\text{error}</td>
<td>&lt; 0.5%$</td>
</tr>
<tr>
<td>Proportion of $</td>
<td>\text{error}</td>
<td>\in [0.5%;1%]$</td>
</tr>
<tr>
<td>Proportion of $</td>
<td>\text{error}</td>
<td>\in [1%;2%]$</td>
</tr>
<tr>
<td>Proportion of $</td>
<td>\text{error}</td>
<td>\in [2%;3%]$</td>
</tr>
<tr>
<td>Proportion of $</td>
<td>\text{error}</td>
<td>&gt; 3%$</td>
</tr>
</tbody>
</table>

Notes: The error is measured as the difference between the exact analytical results provided by Proposition 1 and Proposition 2 and their Monte Carlo approximations, and expressed as a percentage of the exact values. Details about the selected sample and the way Monte Carlo simulation is carried out are the same as in the notes of Table 1.
Table 3 – Numerical assessment of the error entailed by Proposition 4 as a function of the width of the time interval $[0,T]$  

<table>
<thead>
<tr>
<th></th>
<th>$T &lt; 0.1$</th>
<th>$T \in [0.1,0.25]$</th>
<th>$T \in [0.25,1]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean absolute value of error</td>
<td>0.14%</td>
<td>0.35%</td>
<td>1.08%</td>
</tr>
<tr>
<td>maximum absolute value of error</td>
<td>0.58%</td>
<td>1.21%</td>
<td>2.97%</td>
</tr>
<tr>
<td>Proportion of $</td>
<td>\text{error}</td>
<td>&lt; 0.5%$</td>
<td>93%</td>
</tr>
<tr>
<td>Proportion of $</td>
<td>\text{error}</td>
<td>\in [0.5%;1%]$</td>
<td>7%</td>
</tr>
<tr>
<td>Proportion of $</td>
<td>\text{error}</td>
<td>\in [1%;2%]$</td>
<td>0%</td>
</tr>
<tr>
<td>Proportion of $</td>
<td>\text{error}</td>
<td>\in [2%;3%]$</td>
<td>0%</td>
</tr>
<tr>
<td>Proportion of $</td>
<td>\text{error}</td>
<td>&gt; 3%$</td>
<td>0%</td>
</tr>
</tbody>
</table>

Notes: same as in Table 1

Table 4 – Numerical assessment of the error entailed by Proposition 3 as a function of the width of the time interval $[0,T]$  

<table>
<thead>
<tr>
<th></th>
<th>$T &lt; 0.1$</th>
<th>$T \in [0.1,0.25]$</th>
<th>$T \in [0.25,1]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean absolute value of error</td>
<td>0.18%</td>
<td>0.41%</td>
<td>1.36%</td>
</tr>
<tr>
<td>maximum absolute value of error</td>
<td>0.71%</td>
<td>1.45%</td>
<td>5.04%</td>
</tr>
<tr>
<td>Proportion of $</td>
<td>\text{error}</td>
<td>&lt; 0.5%$</td>
<td>86%</td>
</tr>
<tr>
<td>Proportion of $</td>
<td>\text{error}</td>
<td>\in [0.5%;1%]$</td>
<td>14%</td>
</tr>
<tr>
<td>Proportion of $</td>
<td>\text{error}</td>
<td>\in [1%;2%]$</td>
<td>0%</td>
</tr>
<tr>
<td>Proportion of $</td>
<td>\text{error}</td>
<td>\in [2%;3%]$</td>
<td>0%</td>
</tr>
<tr>
<td>Proportion of $</td>
<td>\text{error}</td>
<td>&gt; 3%$</td>
<td>0%</td>
</tr>
</tbody>
</table>

Notes: same as in Table 1

References


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