Numerical Rectification of Curves

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Abstract

An efficient numerical technique has been formulated for determining the length of curves with accuracy of order $O \left( 10^{-16} \right)$. The technique is based on the Cauchy integral formula and makes use of an iterative approach for finding out the derivatives of the generating function at the nodes of the Gauss-Legendre $n$-point rule inside the range of integration. The method has been verified by considering some standard examples.

Mathematics Subject Classification: 65D25, 65D30

Keywords: Rectification of curves, Cauchy integral formula, Quadrature rules

1. Introduction

Determining the length of a curve on a plane or in space is known as rectification of the curve. Rectification of curves is an important problem in physics, computer graphics and engineering. Towards the middle of the seventeenth century Hendrik van Heuraet and Pierre de Fermat, the two eminent mathematicians invented independently the formula for rectification of curves on a plane. The formula for rectification of curve given by
\[ l = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx \quad \text{or} \quad l = \int_{a}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \quad \text{or} \quad l = \int_{\theta_0}^{\theta_1} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta \]  

(1)

according as the curve is given by \( y = f(x), a \leq x \leq b \) (the Cartesian form) or \( x = x(t), y = y(t); \alpha \leq t \leq \beta \) (parametric form) or \( r = f(\theta), \theta_0 \leq \theta \leq \theta_1 \) (polar form). It is assumed that the functions under the integral sign have continuous derivatives. Since rectification of curves involves both the differentiation and integration processes and as such it is sometimes a difficult task.

Significant research work has been undertaken by different researchers in connection with the rectification of curves and its applications (cf Moll, Nowalsky and Solanilla(2002), Chen and Dillen (2005), Ilarslan and Nesovic (2007) and (2008), Abbasi, Olyaee and Jhafari (2013) etc.). Our object in this paper is to devise a numerical technique to determine accurately the length of a curve on a plane.

2. Formulation of the technique

The technique is based on the Cauchy integral formula of complex analysis according to which the derivative of an analytic function \( g(z) \) at a point \( \zeta \) inside a closed contour \( \Gamma \) contained in the domain of analyticity of the function \( g(z) \) is given by

\[ g' (\zeta) = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(z)}{(z - \zeta)^2} \, dz. \]  

(2)

We consider, for the sake of simplicity, the curve in the Cartesian form. The set of abscissas of the Gauss-Legendre \( n \)-point quadrature rule in the interval \( [a, b] \) is given by

\[ g_j = (a + b)/2 + t_j \, (b - a)/2, j = 1(1)n \]  

(3)

where \( t_j \)'s are the zeros of the Legendre polynomial of degree \( n \). Let \( T_j \) be an equilateral triangle with centroid at \( g_j \) and vertices at the following set of points:

\[ z_1 = g_j + s/\sqrt{3} - is, \quad z_2 = g_j + s/\sqrt{3} + is, \quad z_3 = g_j - 2s/\sqrt{3} \]  

(4)
where \( s \) is a small positive number. Replacing the function \( g(z) \) by the function \( f(z) \), the point \( \zeta \) by \( g_j \) and the contour \( \Gamma \) by the equilateral triangle \( T_j \) the equation (2) leads to the following:

\[
f'(g_j) = \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{(z - g_j)^2} dz.
\]

The triangle \( T_j \) consists of the directed line segments \( L_1^{(j)}, L_2^{(j)}, L_3^{(j)} \) from the point \( z_1^{(j)} \) to \( z_2^{(j)} \), \( z_2^{(j)} \) to \( z_3^{(j)} \) and \( z_3^{(j)} \) to \( z_1^{(j)} \) respectively. Since \( T_j = L_1^{(j)} \cup L_2^{(j)} \cup L_3^{(j)} \), from equation (5) we have the following:

\[
f'(g_j) = \frac{1}{2\pi i} \sum_{j=1}^{3} \int_{L_j} \frac{f(z)}{(z - g_j)^2} dz.
\]

The three integrals in equation (6) can be approximated by an open quadrature rule (cf. Milovanovic(2012)) meant for the approximation of the integral of an analytic function \( \psi(z) \) along a directed line segment \( L \) from the point \( z_0 - H \) to the point \( z_0 + H \). For the sake of simplicity and accuracy the transformed \( m \)-point Gauss-Legendre rule due to Lether(1974) is considered to be preferable which is stated as follows:

\[
\int_L \psi(z) dz \approx Q_m(\psi, L) = H \sum_{k=1}^{m} c_k \psi(z_0 + Ht_k)
\]

where \( c_k \)'s are the coefficients. To apply the transformed Gauss-Legendre rule to the integrals in equation (6) along \( L_1^{(j)}, L_2^{(j)}, L_3^{(j)} \) the following three pairs of complex numbers \( z_0, H \) are necessary.

\[
\begin{align*}
z_{10}^{(j)} &= \left( z_2^{(j)} + z_1^{(j)} \right)/2, & H_1^{(j)} &= \left( z_2^{(j)} - z_1^{(j)} \right)/2, \\
z_{20}^{(j)} &= \left( z_3^{(j)} + z_2^{(j)} \right)/2, & H_2^{(j)} &= \left( z_3^{(j)} - z_2^{(j)} \right)/2, \\
z_{30}^{(j)} &= \left( z_1^{(j)} + z_3^{(j)} \right)/2, & H_3^{(j)} &= \left( z_1^{(j)} - z_3^{(j)} \right)/2.
\end{align*}
\]  

For the purpose of fair accuracy of computation the real number \( s \) is assigned reasonably small positive values and the principal part in the Laurent’s expansion of the integrand in equation (6) is subtracted from \( f(z) \). Therefore equation (6) leads to the following:
\[ \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{(z - g_j)^2} \, dz = \frac{1}{2\pi i} \sum_{k=1}^{3} \int_{L_k} \frac{f(z) - f(g_j) + (z - g_j)f'(g_j)}{(z - g_j)^2} \, dz + f'(g_j) \quad (9) \]

Since direct application of the rule \( Q_m(\psi, L) \) given by equation (7) cannot be made for the approximation of the integrals in the right hand side of equation (9) the following iterative procedure is adopted.

\[
\begin{align*}
\delta(\mu + 1) &= \frac{1}{2\pi i} \sum_{k=1}^{3} Q_m(\theta_j, L_k) + \delta(\mu), \\
\theta_j(z) &= \frac{f(z) - f(g_j) - (z - g_j)\delta(\mu)}{(z - g_j)^2} \quad \mu = 1, 2, 3, \ldots \quad (10)
\end{align*}
\]

\( \delta(\mu) \) is the \( \mu \)th iterate. If the initial guess \( \delta(1) = 0 \), then the next iterate \( \delta(2) \) is a reasonable approximation to \( f'(g_j) \). The accuracy of the approximation is improved for values of \( \mu = 2, 3, 4, \ldots \) as the iteration scheme given by equation (10) is clearly convergent to \( f'(g_j) \). The iteration is terminated as soon as two consecutive iterates agree up to fifteen decimal places.

After determining the derivatives for \( j = 1, 2, 3, \ldots, n \) the values of the function \( \eta(x) = \sqrt{1 + [f'(x)]^2} \) at \( g_j, j = 1, 2, 3, \ldots, n \) are computed. Finally the Gauss-Legendre \( n \)-point rule meant for the interval \([a, b]\) is applied which yields the following approximation for the arc length

\[ l = \frac{b - a}{2} \sum_{j=1}^{n} c_j \eta(g_j) \quad (11) \]

3. Some examples and application of the technique

For the numerical tests three curves in different forms have been considered and these are appended in column-1 of Table-1. The parameter \( s \) is assigned the value 0.14. The order of the transformed Gauss-Legendre rule is taken as \( m=6 \) in equation (10) and the order of the Gauss-Legendre rule in equation (11) is taken as \( n=12 \). The abscissas and coefficient can be found in Abramowitz and Stegun (1964). It is noted that the iteration process yields 15 decimal place accuracy in maximum 8 iterations. By modifying the program for execution over the machine, the arc length of 3D curves can be computed by applying of the technique.
Numerical rectification of curves

Table 1

<table>
<thead>
<tr>
<th>Curves</th>
<th>s</th>
<th>N</th>
<th>Exact length</th>
<th>Approximate length</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = \log(\sec x); 0 \leq x \leq \pi/4$</td>
<td>0.14</td>
<td>6</td>
<td>0.881373587019543</td>
<td>0.881373587019543</td>
</tr>
<tr>
<td>$x(t) = t^2, y(t) = t - t^3/3; \quad 0 \leq t \leq \sqrt{3}$</td>
<td>0.14</td>
<td>6</td>
<td>3.464101615137754</td>
<td>3.464101615137754</td>
</tr>
<tr>
<td>$r = 1 - \cos \theta; 0 \leq \theta \leq 2\pi$</td>
<td>0.14</td>
<td>8</td>
<td>4.000000000000000</td>
<td>4.000000000000000</td>
</tr>
</tbody>
</table>

References


Received: September 15, 2013