A Pointwise Approximation for Random Sums of Independent Discrete Random Variables

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Abstract

We determine a new result of pointwise Poisson approximation for random sums of independent integer-valued random variables. The bound in this study is sharper than that reported in [1].

Mathematics Subject Classification: 60F05, 60G05

Keywords: Integer-valued random variable, Pointwise Poisson approximation, Random sums

1 Introduction

Let \( \{X_{n,i}, i = 1, \ldots, n; n \in \mathbb{N}\} \) be a sequence of independent integer-valued random variables with success probabilities \( P(X_{n,i} = 1) = p_{n,i} \) and \( P(X_{n,i} = 0) = 1 - p_{n,i} - q_{n,i} \) for every \( i \in \{1, \ldots, n\} \) and \( n \in \mathbb{N} \), where \( p_{n,i}, q_{n,i} \in (0, 1) \) and \( p_{n,i} + q_{n,i} \in (0, 1) \). Suppose that \( N_n \) for \( n \in \mathbb{N} \) are positive integer-valued random variables and independent of the \( X_{n,i} \)'s. Let \( S_{N_n} = \sum_{i=1}^{N_n} X_{n,i} \) be random sums of a sequence of independent integer-valued random variables, and let \( \lambda_{N_n} = \sum_{i=1}^{N_n} p_{n,i} \). In this case, Hung and Giang [1] gave a pointwise bound for approximating the probability function of the random sums \( S_{N_n} \) by the Poisson probability function with mean \( \lambda_{N_n} \) as follows:

\[
d_P(S_{N_n}, \varphi_{\lambda_{N_n}}) \leq 2E \left\{ \sum_{i=1}^{N_n} (p_{N_n,i}^2 + q_{N_n,i}) \right\},
\]

(1.1)
where \( d_P(S_{N_n}, \varphi_{\lambda N_n}) = |P(S_{N_n} = x) - P(\varphi_{\lambda N_n} = x)| \) for \( x \in \mathbb{N} \cup \{0\} \) and \( \varphi_{\lambda N_n} \) is the Poisson random variable with mean \( \lambda N_n \).

In this paper, we determine a new bound for \( d_P(S_{N_n}, \varphi_{\lambda N_n}) \) by using the method in [3], which is in Section 2. In Section 3, we determine a new bound for this approximation and the conclusion of this study is presented in the last section.

2 Method

We will prove our main result by using the method in [3]. For arbitrary non-negative integer-valued \( X, Y \) and \( Z \) and \( A \subseteq \mathbb{N} \cup \{0\} \),

\[
|P(X \in A) - P(Y \in A)| \leq |P(X \in A) - P(Z \in A)| \\
+ |P(Z \in A) - P(Y \in A)|. \tag{2.1}
\]

The following lemmas are directly obtained from [3] and [2], respectively.

**Lemma 2.1.** Let \( X'_{n,i} = I(X_{n,i} = 1) \) for every \( i \in \{1, ..., n\} \) and \( S'_n = \sum_{i=1}^{n} X'_{n,i} \). Then, for \( x \in \mathbb{N} \cup \{0\} \),

\[
d_P(S_n, S'_n) \leq \sum_{i=1}^{n} q_{n,i} \tag{2.2}
\]

and

\[
d_P(S'_n, \varphi_{\lambda_n}) \leq \lambda_n^{-1}(1 - e^{-\lambda_n}) \sum_{i=1}^{n} p_{n,i}^2. \tag{2.3}
\]

3 Result

This Section presents a new bound for \( d_P(S_{N_n}, \varphi_{\lambda N_n}) \).

**Theorem 3.1.** With the above definitions, the following inequality holds:

\[
d_P(S_{N_n}, \varphi_{\lambda N_n}) \leq E \left\{ \sum_{i=1}^{N_n} (p_{N_n,i}^2 + q_{N_n,i}) \right\} \tag{3.1}
\]

for \( x \in \mathbb{N} \cup \{0\} \).
Proof. From (2.1), (2.2) and (2.3), we obtain
\[ d_P(S_{N_n}, \varphi_{\lambda_{N_n}}) \leq \sum_{m=1}^{\infty} P(N_n = m) d_P(S_m, \varphi_{\lambda_m}) \]
\[ \leq \sum_{m=1}^{\infty} P(N_n = m) \left[ d_P(S_m', \varphi_{\lambda_m}) + d_P(S_m, S_m') \right] \]
\[ \leq \sum_{m=1}^{\infty} P(N_n = m) \left[ \lambda_m^{-1} (1 - e^{-\lambda_m}) \sum_{i=1}^{m} p_{m,i}^2 + \sum_{i=1}^{m} q_{m,i} \right] \]
\[ \leq \sum_{m=1}^{\infty} P(N_n = m) \sum_{i=1}^{m} (p_{m,i}^2 + q_{m,i}) \]
which implies that
\[ d_P(S_{N_n}, \varphi_{\lambda_{N_n}}) \leq E \left\{ \sum_{i=1}^{N_n} (p_{N_n,i}^2 + q_{N_n,i}) \right\}. \]
Hence, (3.1) is obtained. \( \Box \)

Remark. Because \( E \left\{ \sum_{i=1}^{N_n} (p_{N_n,i}^2 + q_{N_n,i}) \right\} < 2E \left\{ \sum_{i=1}^{N_n} (p_{N_n,i}^2 + q_{N_n,i}) \right\}, \) the bound in (3.1) is sharper than the bound in (1.1).

4 Conclusion

A new result for approximating the probability function of random sums of independent integer-valued random variables and the Poisson probability function with mean \( \lambda_{N_n} = \sum_{i=1}^{N_n} p_{n,i} \) was obtained. By comparing between two such bounds, the bound in this study is sharper than that reported in [1].

References


Received: October 11, 2014; Published: December 1, 2014