Static Model of Decision-making
over the Set of Coalitional Partitions

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Abstract

Let be $N$ the set of players and $M$ the set of projects. The coalitional model of decision-making over the set of projects is formalized as family of games with different fixed coalitional partitions for each project that required the adoption of a positive or negative decision by each of the players. The players’ strategies are decisions about each of the project. Players can form coalitions in order to obtain higher income. Thus, for each project a coalitional game is defined. In each coalitional game it is required to find in some sense optimal solution. Solving successively each of the coalitional games, we get the set of optimal $n$-tuples for all coalitional games. It is required to find a compromise solution for the choice of a project, i.e. it is required to find a compromise coalitional partition. As an optimality principles are accepted generalized PMS-vector [1], [2] and its modifications, and compromise solution.

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1 Introduction

The set of agents $N$ and the set of projects $M$ are given. Each agent fixed his participation or not participation in the project by one or zero choice. The participation in the project is connected with incomes or losses which the
agents wants to maximize or minimize. Agents may form coalitions. This gives us an optimization problem which can be modeled as a game. This problem we call as static coalitional model of decision-making.

Denote the players by \( i \in N \) and the projects by \( j \in M \). The family \( M \) of different games are considered. In each game \( G_j, j \in M \) the player \( i \) has two strategies accept or reject the project. The payoff of the player in each game is determined by the strategies chosen by all players in this game \( G_j \). As it was mentioned before the players can form coalitions to increase the payoffs. In each game \( G_j \) coalitional partition is formed and the problem is to find the optimal strategies for coalitions and the imputation of the coalitional payoff between the members of the coalition. The games \( G_1, \ldots, G_m \) are solved by using the PMS-vector [1], [2] and its modifications.

Then having the solutions of games \( G_j, j = 1, m \) the new optimality principle - “the compromise solution” is proposed to select the best projects \( j^* \in M \).

2 State of the problem

Consider the following problem. Suppose:

\( N = \{1, \ldots, n\} \) is the set of players;

\( X_i = \{0; 1\} \) is the set of pure strategies \( x_i \) of player \( i, i = 1, n \). The strategy \( x_i \) can take the following values: \( x_i = 0 \) as a negative decision for some project and \( x_i = 1 \) as a positive decision;

\( l_i = 2 \) is the number of pure strategies of player \( i \);

\( x \) is the \( n \)-tuple of pure strategies chosen by the players;

\( X = \prod_{i=1}^{n} X_i \) is the set of \( n \)-tuples;

\( \mu_i = (\xi_i^0, \xi_i^1) \) is the mixed strategy of player \( i \), where \( \xi_i^0 \) is the probability of making negative decision by the player \( i \) for some project, and \( \xi_i^1 \) is the probability of making positive decision correspondingly;

\( M_i \) is the set of mixed strategies of the \( i \)-th player;

\( \mu \) is the \( n \)-tuple of mixed strategies chosen by players for some project;

\( M = \prod_{i=1}^{n} M_i \) is the set of \( n \)-tuples in mixed strategies for some project;

\( K_i (x) : X \rightarrow R^1 \) is the payoff function defined on the set \( X \) for each player \( i, i = 1, n \), and for some project.

Thus, for some project we have noncooperative \( n \)-person game \( G(x) \):

\[
G(x) = \langle N, \{X_i\}_{i=1,n}, \{K_i (x)\}_{i=1,n,x \in X} \rangle.
\] (1)

Now suppose \( M = \{1, \ldots, m\} \) is the set of projects, which require making positive or negative decision by \( n \) players.
A coalitional partitions $\Sigma^j$ of the set $N$ is defined for all $j = 1, m$:

$$\Sigma^j = \left\{ S_1^j, \ldots, S_l^j \right\}, \ l \leq n, \ n = |N|, \ S_k^j \cap S_q^j = \emptyset \ \forall \ k \neq q, \ \bigcup_{k=1}^{l} S_k^j = N.$$ 

Then we have $m$ simultaneous $l$-person coalitional games $G_j (x_{\Sigma^j}), j = 1, m$, in a normal form associated with the respective game $G(x)$:

$$G_j (x_{\Sigma^j}) = \left\langle N, \left\{ \bar{X}_{S_k^j} \right\}_{k=1, l, S_k^j \in \Sigma^j}, \left\{ \bar{H}_{S_k^j} (x_{\Sigma^j}) \right\}_{k=1, l, S_k^j \in \Sigma^j} \right\rangle, \ j = 1, m. \quad (2)$$

Here for all $j = 1, m$:

$$\bar{x}_{S_k^j} = \left\{ x_i \right\}_{i \in S_k^j} \text{ is the } l\text{-tuple of strategies of players from coalition } S_k^j, \ k = 1, l;$$

$$\bar{X}_{S_k^j} = \prod_{i \in S_k^j} X_i$$

is the set of strategies $\bar{x}_{S_k^j}$ of coalition $S_k^j, k = 1, l$, i.e. Cartesian product of the sets of players’ strategies, which are included into coalition $S_k^j$:

$$x_{\Sigma^j} = (\bar{x}_{S_1^j}, \ldots, \bar{x}_{S_l^j}) \in \bar{X}, \ \bar{x}_{S_k^j} \in \bar{X}_{S_k^j}, \ k = 1, l \text{ is the } l\text{-tuple of strategies of all coalitions};$$

$$\bar{X} = \prod_{k=1, l} \bar{X}_{S_k^j}$$

is the set of $l$-tuples in the game $G_j (x_{\Sigma^j})$;

$$l_{S_k^j} = |\bar{X}_{S_k^j}| = \prod_{i \in S_k^j} l_i$$

is the number of pure strategies of coalition $S_k^j$;

$$l_{\Sigma^j} = \prod_{k=1, l} l_{S_k^j}$$

is the number of $l$-tuples in pure strategies in the game $G_j (x_{\Sigma^j})$.

$\bar{M}_{S_k^j}$ is the set of mixed strategies $\bar{\mu}_{S_k^j}$ of the coalition $S_k^j, k = 1, l$;

$$\bar{\mu}_{S_k^j} = \left( \bar{\mu}_{S_k^j}^1, \ldots, \bar{\mu}_{S_k^j}^l \right), \ \bar{\mu}_{S_k^j}^\xi \geq 0, \ \xi = 1, l_{S_k^j}, \ \sum_{\xi=1}^{l_{S_k^j}} \bar{\mu}_{S_k^j}^\xi = 1, \text{ is the mixed strategy, that is the set of mixed strategies of players from coalition } S_k^j, k = 1, l;$$

$$\mu_{\Sigma^j} = (\bar{\mu}_{S_1^j}, \ldots, \bar{\mu}_{S_l^j}) \in \bar{M}, \ \bar{\mu}_{S_k^j} \in \bar{M}_{S_k^j}, k = 1, l, \text{ is the } l\text{-tuple of mixed strategies};$$

$$\bar{M} = \prod_{k=1, l} \bar{M}_{S_k^j}$$

is the set of $l$-tuples in mixed strategies.

From the definition of strategy $\bar{x}_{S_k^j}$ of coalition $S_k^j$ it follows that $x_{\Sigma^j} = (\bar{x}_{S_1^j}, \ldots, \bar{x}_{S_l^j})$ and $x = (x_1, \ldots, x_n)$ are the same $n$-tuples in the games $G(x)$ and $G_j (x_{\Sigma^j})$. However it does not mean that $\mu = \mu_{\Sigma^j}$.

Payoff function $\bar{H}_{S_k^j} : \bar{X} \to R^1$ of coalition $S_k^j$ for the fixed projects $j, j = 1, m$, and for the coalitional partition $\Sigma^j$ is defined under condition
that:

\[ \bar{H}_{S_k^j}(x_{\Sigma^j}) \geq H_{S_k^j}(x_{\Sigma^j}) = \sum_{i \in S_k^j} K_i(x), \quad k = \overline{1, l}, \quad j = \overline{1, m}, \quad S_k^j \in \Sigma^j, \quad (3) \]

where \( K_i(x), \ i \in S_k^j, \) is the payoff function of player \( i \) in the \( n \)-tuple \( x_{\Sigma^j} \).

**Definition 1.** A set of \( m \) coalitional \( l \)-person games defined by (2) is called *static coalitional model of decision-making*.

**Definition 2.** *Solution of the static coalitional model of decision-making in pure strategies* is \( x^*_{\Sigma^j} \), that is Nash equilibrium (NE) in a pure strategies in \( l \)-person game \( G_{j^*}(x_{\Sigma^j}) \), with the coalitional partition \( \Sigma^{j^*} \), where coalitional partition \( \Sigma^{j^*} \) is the compromise coalitional partition (see 2.2).

**Definition 3.** *Solution of the static coalitional model of decision-making in mixed strategies* is \( \mu^*_{\Sigma^j} \), that is Nash equilibrium (NE) in a mixed strategies in \( l \)-person game \( G_{j^*}(\mu_{\Sigma^j}) \), with the coalitional partition \( \Sigma^{j^*} \), where coalitional partition \( \Sigma^{j^*} \) is the compromise coalitional partition (see 2.2).

Generalized PMS-vector is used as the coalitional imputation [1], [2].

### 3 Algorithm for solving the problem

#### 3.1 Algorithm of constructing the generalized PMS-vector in a coalitional game.

Remind the algorithm of constructing the generalized PMS-vector in a coalitional game [1], [2].

1. Calculate the values of payoff \( \bar{H}_{S_k^j}(x_{\Sigma^j}) \) for all coalitions \( S_k^j \in \Sigma^j, \ k = \overline{1, l} \), for coalitional game \( G_{j^*}(x_{\Sigma^j}) \) by using formula (3).

2. Find NE [3] \( x^*_{\Sigma^j} \) or \( \mu^*_{\Sigma^j} \) (one or more) in the game \( G_{j^*}(x_{\Sigma^j}) \). The payoffs’ vector of coalitions in NE in mixed strategies \( E(\mu^*_{\Sigma^j}) = \{v(S_k^j)\}_{k=1}^{l} \).

Denote a payoff of coalition \( S_k^j \) in NE in mixed strategies by

\[ v(S_k^j) = \sum_{\tau=1}^{l_{\Sigma^j}} p_{r,j} \bar{H}_{r,S_k^j}(x_{\Sigma^j}^*), k = \overline{1, l_{\Sigma^j}}, \]

where

- \( \bar{H}_{r,S_k^j}(x_{\Sigma^j}^*) \) is the payoff of coalition \( S_k^j \), when coalitions choose their pure strategies \( x_{\Sigma^j}^* \) in NE in mixed strategies \( \mu^*_{\Sigma^j} \).

- \( p_{r,j} = \prod_{k=1}^{l} \bar{\mu}_{S_k^j}, \ \xi_k = \overline{1, l_{S_k^j}}, \ \tau = \overline{1, l_{\Sigma^j}}, \) is probability of the payoff’s realization \( \bar{H}_{r,S_k^j}(x_{\Sigma^j}^*) \) of coalition \( S_k^j \).


The value $\bar{H}_{\tau,S_j} (x^*_\Sigma)$ is random variable. There could be many $l$-tuple of NE in the game, therefore, $v \left(S^i_j\right)$, $\ldots$, $v \left(S^j_i\right)$, are not uniquely defined.

The payoff of each coalition in NE $E \left(\mu^*_j\right)$ is divided according to Shapley’s value $[4]$ $Sh \left(S^k_j\right) = \left(Sh \left(S^j_k : 1\right), \ldots, Sh \left(S^j_k : s\right)\right)$:

$$Sh \left(S^j_k : i\right) = \sum_{s' \subseteq S^j_k \atop s' \ni i} \frac{(s'-1)! (s-s')!}{s!} [v \left(S'\right) - v \left(S' \setminus \{i\}\right)] \quad \forall \ i = 1, s, \quad (4)$$

where $s = |S^j_k|$ $(s' = |S'|)$ is the number of elements of sets $S^j_k \ (S')$, and $v \left(S'\right)$ are the total maximal guaranteed payoffs all over the $S' \subset S_k$.

Moreover

$$v \left(S^j_k\right) = \sum_{i=1}^{s} Sh \left(S^j_k : i\right).$$

Then PMS-vector in the NE in mixed strategies $\mu^*_j$ in the game $G_j(x^*_\Sigma)$ is defined as

$$PMS^j (\mu^*_j) = \left(PMS^j_1 (\mu^*_j), \ldots, PMS^j_n (\mu^*_j)\right),$$

where

$$PMS^j_i (\mu^*_j) = Sh \left(S^j_k : i\right), \ i \in S^j_k, \ k = 1, l.$$

3.2 Algorithm for finding a set of compromise solutions.

We also remind the algorithm for finding a set of compromise solutions ([5]; p.18).

$$C_{PMS} (M) = \arg \min_j \max_i \left\{\max_j PMS^j_i - PMS^j_i\right\}.\quad (5)$$

**Step 1.** Construct the ideal vector $R = (R_1, \ldots, R_n)$, where $R_i = PMS^j_i = \max_j PMS^j_i$ is the maximal value of payoff’s function of player $i$ in NE on the set $M$, and $j$ is the number of project $j \in M$:

$$\begin{pmatrix}
PMS^1_1 & \ldots & PMS^1_n \\
\vdots & \ddots & \vdots \\
PMS^m_1 & \ldots & PMS^m_n \\
\downarrow & \ddots & \downarrow \\
PMS^j_1 & \ldots & PMS^j_n
\end{pmatrix}$$

**Step 2.** For each $j$ find deviation of payoff function values for other players from the maximal value, that is $\Delta^j_i = R_i - PMS^j_i$, $i = 1, n$:

$$\Delta = \begin{pmatrix}
R_1 - PMS^1_1 & \ldots & R_n - PMS^1_n \\
\vdots & \ddots & \vdots \\
R_1 - PMS^m_1 & \ldots & R_n - PMS^m_n
\end{pmatrix}.$$
Step 3. From the found deviations $\Delta^j_i$ for each $j$ select the maximal deviation $\Delta^j_i = \max_i \Delta^j_i$ among all players $i$:

$$
\begin{pmatrix}
R_1 - \text{PMS}^1_1 & \ldots & R_n - \text{PMS}^1_n \\
\vdots & \ddots & \vdots \\
R_1 - \text{PMS}^m_1 & \ldots & R_n - \text{PMS}^m_n
\end{pmatrix}
= 
\begin{pmatrix}
\Delta^1_1 & \ldots & \Delta^1_n \\
\vdots & \ddots & \vdots \\
\Delta^m_1 & \ldots & \Delta^m_n
\end{pmatrix}
\rightarrow 
\Delta^1_i^j 

\rightarrow 
\Delta^m_i^j
$$

Step 4. Choose the minimal deviation for all $j$ from all the maximal deviations among all players $i$ $\Delta^{j^*}_{i^*} = \min_j \Delta^j_i = \min \max_i \Delta^j_i$.

The project $j^* \in C_{\text{PMS}} (M)$, on which the minimum is reached is a compromise solution of the game $G_j(x_{\Sigma^j})$ for all players.

3.3 Algorithm for solving the static coalitional model of decision-making.

Thus, we have an algorithm for solving the problem.

1. Fix a $j$, $j = 1, m$.
2. Find the NE $\mu^*_{\Sigma^j}$ in the coalitional game $G_j(x_{\Sigma^j})$ associated with the noncooperative game $G(x)$ and find imputation in NE, that is $\text{PMS}^j(\mu^*_{\Sigma^j})$.
3. Repeat iterations 1-2 for all other $j$, $j = 1, m$.
4. Find compromise solution $j^*$, that is $j^* \in C_{\text{PMS}} (M)$.

4 Conclusion

A static coalitional model of decision-making and algorithm for finding optimal solution are constructed in this paper.

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