On the Appropriate Choice of the Risk Model

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Dedicated to Professor Marius Stoka on the occasion of his 80th birthday

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Abstract

Recently the classical actuarial risk model for the evaluation of the ruin probability has been generalized to include dependency relations between the claim occurrences times and the claim amounts. For instance it has been incorporated into the stochastic model that the distributions of the times between consecutive claim occurrences times may depend on the last previous claim amount [1] or that claim sizes depend on the past of the point process of instants when claims are presented [2,3]. Results on ruin probabilities and related quantities have been published in several papers under such assumptions. This evolution implies that in a concrete application we have to choose between different versions of the actuarial risk model. This choice should be performed in a reasonable and objective way taking into account our empirical knowledge of the risk process. This leads us naturally to develop statistical tests able to distinguish certain classes of marked point processes. In order to assure the optimal use of the data it seems indicated to look for locally most powerful tests [4] and a clever choice of the period of observation of the risk process greatly facilitates the task and leads to simple procedures which are easy to implement.

Keywords: Actuarial risk models, dependencies between claim occurrences times and claim amounts, (locally) most powerful statistical tests, embedded renewal processes, choice of the period of observation of the risk process

1. Introduction

Recently the classical risk models for the evaluation of ruin probabilities have been generalized to include dependencies between claim occurrence times and claim sizes. Several authors have developed results on ruin probabilities and related relevant quantities under these less restrictive conditions.

In view of these extensions the need arises to choose in concrete applications in a reasonable and objective way between the classical models and their generalizations. Since the stochastic part of the risk process can be described by a marked point process the choice among different ruin models amounts to perform a test discriminating different classes of marked point processes based on the observation of the investigated risk process in a given time interval.

To illustrate these ideas we present in the sequel statistical tests opposing the classical compound Poisson ruin model and one of four generalizations of it introduced in the scientific literature.

The following notation is used in this article:

We designate random variables by capital letters and their realized values by the corresponding lower case letters. Let $T(i)$ be the occurrence time of the i th claim after the instant $t = 0$ and $B(i)$ the associated claim size. We denote by $U(i) := T(i) - T(i-1)$ [$i = 1, 2, \dots$] (with the convention $P(T(0) = 0) = 1$) the i th gap between consecutive point events. Φ stands for the distribution function of the standard normal distribution, $Exp(\lambda)$ for the exponential distribution with parameter λ [$\lambda > 0$] and $\sim$ means "distributed according to". R^+ is the set of positive real numbers, $\mathcal{N}$ the set of natural numbers and the function 1_A is given by $1_A(x) = 1$ for $x \in A$ and $1_A(x) = 0$ for $x \notin A$.

2. Should the risk model of Albrecher and Boxma be used?

In [1, Model 1 and 3] Albrecher and Boxma propose a ruin model with the property that if the claim $B(i)$ is larger than the random threshold $R(i)$ then the time until the next claim is independent exponentially distributed with parameter λ_1 , otherwise it is independent exponentially distributed with parameter λ_2 . The quantities $R(i)$ are assumed to be independent identically distributed with distribution function F_R and the claim sizes $B(i)$ are supposed to be independent, identically distributed with density function f_B , both distributions concentrated on R^+ and independent. We choose as observation period of the investigated risk process the interval $(0, t_{n_0}]$ from $t = 0$ to the occurrence time of the n_0 th claim [$n_0 \in \mathcal{N}$]. Our observation of the associated marked point process is then specified by the data $(t_1, b_1; \dots; t_{n_0}, b_{n_0})$. We seek a locally most powerful test [4] of $H_0 : g = 0$ versus $H_1 : g > 0$,

choosing $\lambda_1 = \lambda + g$ $\lambda_2 = \lambda$ [$\lambda, g > 0$] and assuming that the initial condition $U(1) \sim \text{Exp}(\lambda)$ is satisfied. The likelihood function is given by

$$L(\lambda, f_B, F_R, g : t_1, b_1; \dots; t_{n_0}, b_{n_0}) = \lambda e^{-\lambda u_1} f_B(b_1) \cdot [1_{\{B(1) \leq R(1)\}}(b_1) \lambda e^{-\lambda u_2} + 1_{\{B(1) > R(1)\}}(b_1) (\lambda + g) e^{-(\lambda+g)u_2}] f_B(b_2) \cdot \dots \cdot [1_{\{B(n_0-1) \leq R(n_0-1)\}}(b_{n_0-1}) \lambda e^{-\lambda u_{n_0}} + 1_{\{B(n_0-1) > R(n_0-1)\}}(b_{n_0-1}) (\lambda + g) e^{-(\lambda+g)u_{n_0}}] f_B(b_{n_0}).$$

For the log likelihood function results

$$\ln(L(\lambda, f_B, F_R, g : t_1, b_1; \dots; t_{n_0}, b_{n_0})) = \sum_{i=1}^{n_0} \ln(f_B(b_i)) + \ln(\lambda e^{-\lambda u_1}) + \sum_{i=2}^{n_0} \ln[1_{\{B(i-1) \leq R(i-1)\}}(b_{i-1}) \lambda e^{-\lambda u_i} + 1_{\{B(i-1) > R(i-1)\}}(b_{i-1}) (\lambda + g) e^{-(\lambda+g)u_i}],$$

and for its derivative near the null hypothesis

$$\partial \ln(L(\lambda, f_B, F_R, g : t_1, b_1; \dots; t_{n_0}, b_{n_0})) / \partial g|_{g=0} = \sum_{i=2}^{n_0} [1_{\{B(i-1) > R(i-1)\}}(b_{i-1}) [e^{-\lambda u_i} - \lambda e^{-\lambda u_i} u_i] / (\lambda e^{-\lambda u_i})].$$

Thus we find as score statistic [4]

$$T_{\lambda, f_B, F_R; n_0}(0) = \sum_{i=2}^{n_0} [1_{\{B(i-1) > R(i-1)\}}(B(i-1)) (1/\lambda - U(i))].$$

We obtain under H_0

$$E[T_{\lambda, f_B, F_R; n_0}(0) : g = 0] = \sum_{i=2}^{n_0} E[1_{\{B(i-1) > R(i-1)\}}(B(i-1)) (1/\lambda - 1/\lambda)] = 0$$

and

$$\text{Var}[T_{\lambda, f_B, F_R; n_0}(0) : g = 0] = \sum_{i=2}^{n_0} E[1_{\{B(i-1) > R(i-1)\}}(B(i-1)) \text{Var}[U(i) : g = 0]] = (n_0 - 1) [\int_0^\infty F_R(b) f_B(b) db] / \lambda^2.$$

Under H_0 the additive terms of $T_{\lambda, f_B, F_R; n_0}(0)$ are independent, identically distributed. According to the standard central limit theorem we have

$$[(n_0 - 1) \int_0^\infty F_R(b) f_B(b) db]^{-1/2} \lambda T_{\lambda, f_B, F_R; n_0}(0) \sim \Phi \text{ for } n_0 \rightarrow \infty.$$

The critical region of the test of size α and for large n_0 is formed by the large values of the standardized score statistic and specified by the condition

$$[(n_0 - 1) \int_0^\infty F_R(b) f_B(b) db]^{-1/2} \lambda t_{\lambda, f_B, F_R; n_0}(0) > z(1 - \alpha)$$

with $\Phi(z(1 - \alpha)) = 1 - \alpha$.

For the special case of a deterministic threshold R^* (i.e. $P(R = R^*) = 1$) we get $\int_0^\infty F_R(b) f_B(b) db = \int_{R^*}^\infty f_B(b) db = 1 - F_B(R^*)$, where F_B denotes the distribution function of B .

3. Should the risk model of Boudreault et alia be used?

In [2] Boudreault, Cossette, Landriault and Marceau propose a risk model where in contrast to the classical compound Poisson model the distribution of the next claim depends on the time elapsed since the last claim, i.e. $B(k)|U(k)$ follows the density function

$$f_{B(k)|U(k)}(b_k|u_k : g) = e^{-gu_k} f_1(b_k) + (1 - e^{-gu_k}) f_2(b_k) [k \in \mathcal{N}],$$

where f_1 and f_2 are known density functions on R^+ . We check whether the

assumptions of this stochastic model are realistic for the observed data in the period $(0, t_{n_0}]$ by means of a locally most powerful test of the hypotheses $H_0 : g = 0$ and $H_1 : g > 0$. The likelihood function reads

$$L(\lambda, f_1, f_2, g : t_1, b_1; \dots; t_{n_0}, b_{n_0}) = \prod_{i=1}^{n_0} [\lambda e^{-\lambda u_i} (e^{-g u_i} f_1(b_i) + (1 - e^{-g u_i}) f_2(b_i))].$$

Derivation of the log likelihood function with respect to g at $g = 0$ yields the score statistic

$$T_{\lambda, f_1, f_2; n_0}(0) = \sum_{i=1}^{n_0} [U(i)(\{f_2(B(i))/f_1(B(i))\} - 1)].$$

This statistic has under H_0 mean 0 and variance

$$\text{Var}[T_{\lambda, f_1, f_2; n_0}(0) : g = 0] = (2n_0/\lambda^2) \int_0^\infty [(f_2(b) - f_1(b))^2 / f_1(b)] db.$$

If this quantity is finite and different from 0, then according to the standard central limit theorem

$$[(2n_0/\lambda^2) \int_0^\infty [(f_2(b) - f_1(b))^2 / f_1(b)] db]^{-1/2}.$$

$$T_{\lambda, f_1, f_2; n_0}(0) \sim \Phi$$

for large n_0 and H_0 is rejected for large values of this standardized test statistic.

4. Should the alternative risk model of Albrecher and Boxma be used?

In [1, Model 2] Albrecher and Boxma propose a risk model where the time until the next claim is exponentially distributed with either parameter λ or $\lambda + g$ [$g > 0$]. If a claim $B(i)$ is smaller or equal than the random threshold $R(i)$ the λ -value of the distribution of the next gap changes from one of these λ -values to the other, if this condition is not satisfied the λ -value stays unaltered. The quantities $R(i)$ are assumed to be independent, identically distributed with distribution F_R and the claim sizes are supposed to be independent identically distributed with density function f_B , both distributions concentrated on R^+ and independent. Let t_{ν_s} be the occurrence time of the sth "small claim" after the instant $t = 0$, where a "small claim" satisfies by definition the condition $\{b_k \leq r_k\}$, while we have $\{b_k > r_k\}$ for $k \in \mathcal{N} - \{\nu_1, \nu_2, \dots\}$. We note that the increments of the claim aggregated process in the periods $(0, t_{\nu_2}]$, $(t_{\nu_2}, t_{\nu_4}]$, $\dots$ are independent identically distributed and form thus a renewal process. For a prefixed $m \in \mathcal{N}$ the choice of $(0, t_{\nu_{2m}}]$ as observation period is adopted, since it leads to a considerable simplification of the procedure. We put $\lambda(1) = \lambda(3) = \dots = \lambda$ and $\lambda(2) = \lambda(4) = \dots = \lambda + g$ for the successively utilised λ -values and check by means of the locally most powerful test of the hypotheses $H_0 : g = 0$ and $H_1 : g > 0$ whether the assumptions underlying model 2 are fulfilled for the observed risk process. The likelihood function is given by

$$L(\lambda, F_R, f_B, g : t_1, b_1; \dots; t_{\nu_{2m}}, b_{\nu_{2m}}) = \lambda e^{-\lambda u_1} f_B(b_1).$$

$$\prod_{k=2}^{\nu_{2m}} [1_{\{B(k-1) > R(k-1)\}}(b_{k-1}) \lambda (1 + \sum_{i=1}^{k-2} 1_{\{B(i) \leq R(i)\}}(b_i)) e^{-\lambda(1 + \sum_{i=1}^{k-2} 1_{\{B(i) \leq R(i)\}}(b_i)) u_k} + 1_{\{B(k-1) \leq R(k-1)\}}(b_{k-1}) \lambda (1 + \sum_{i=1}^{k-1} 1_{\{B(i) \leq R(i)\}}(b_i)) e^{-\lambda(1 + \sum_{i=1}^{k-1} 1_{\{B(i) \leq R(i)\}}(b_i)) u_k}] f_B(b_k),$$

and the log likelihood function takes the form

$$\ln(L(\lambda, F_R, f_B, g : t_1, b_1; \dots; t_{\nu_{2m}}, b_{\nu_{2m}})) = \ln(\lambda e^{-\lambda u_1}) + \sum_{i=1}^{\nu_{2m}} \ln(f_B(b_i)) + \sum_{i=2}^{\nu_{2m}} \ln(N_i),$$

where

$$\ln(N_i) = \ln[1_{\{B(i-1) > R(i-1)\}}(b_{i-1}) \lambda (1 + \sum_{\nu=1}^{i-2} 1_{\{B(\nu) \leq R(\nu)\}}(b_\nu)) e^{-\lambda(1 + \sum_{\nu=1}^{i-2} 1_{\{B(\nu) \leq R(\nu)\}}(b_\nu)) u_i} + 1_{\{B(i-1) \leq R(i-1)\}}(b_{i-1}) \lambda (1 + \sum_{\nu=1}^{i-1} 1_{\{B(\nu) \leq R(\nu)\}}(b_\nu)) e^{-\lambda(1 + \sum_{\nu=1}^{i-1} 1_{\{B(\nu) \leq R(\nu)\}}(b_\nu)) u_i}].$$

If $1 + \sum_{\nu=1}^{i-2} 1_{\{B(\nu) \leq R(\nu)\}}(b_\nu)$ in N_i is uneven, then

$$\partial \ln[N_i] / \partial g|_{g=0} = [\lambda e^{-\lambda u_i}]^{-1} \partial [1_{\{B(i-1) \leq R(i-1)\}}(b_{i-1}) (\lambda + g) e^{-(\lambda+g) u_i}] / \partial g|_{g=0} = 1_{\{B(i-1) \leq R(i-1)\}}(b_{i-1}) ((1/\lambda) - u(i)),$$

and if $1 + \sum_{\nu=1}^{i-2} 1_{\{B(\nu) \leq R(\nu)\}}(b_\nu)$ in N_i is even, then

$$\partial \ln[N_i] / \partial g|_{g=0} = [\lambda e^{-\lambda u_i}]^{-1} \partial [1_{\{B(i-1) > R(i-1)\}}(b_{i-1}) (\lambda + g) e^{-(\lambda+g) u_i}] / \partial g|_{g=0} = 1_{\{B(i-1) > R(i-1)\}}(b_{i-1}) ((1/\lambda) - u(i)).$$

Therefore the test statistic is given by

$$T_{\lambda, F_R, f_B; \nu_{2m}}(0) = \sum_{l=1}^m [1_{\{B(\nu_{2l-1}) \leq R(\nu_{2l-1})\}}(b_{\nu_{2l-1}}) ((1/\lambda) - U(\nu_{2l-1} + 1) + \sum_{\nu=\nu_{2l-1}+1}^{\nu_{2l}-1} 1_{\{B(\nu) > R(\nu)\}}(b_\nu) ((1/\lambda) - U(\nu + 1))].$$

Since under H_0 the $(B(i), R(i))$'s are independent of the $U(i+1)$'s and $E[U(i+1) : H_0] = 1/\lambda$, we find

$$E[T_{\lambda, F_R, f_B; \nu_{2m}}(0) : g = 0] = 0. \text{ For } P(B \leq R) =: p \text{ and given } (\nu_1, \dots, \nu_{2m}) \text{ the probability of the realization of an observation with these } \nu\text{-values is } (1-p)^{\nu_1-1} p (1-p)^{\nu_2-\nu_1-1} \dots (1-p)^{\nu_{2m}-\nu_{2m-1}-1} p.$$

If we subdivide the typical observation in m cycles stretching from $t_{\nu_{2(l-1)}}$ to $t_{\nu_{2l}}$ for $l = 1, \dots, m$ and putting $t_{\nu_0} = 0$ the successive increments of these cycles form an embedded renewal process and the probability of the l th cycle is

$$(1-p)^{\nu_{2l}-\nu_{2(l-1)}-2} p^2.$$

Now $Var[(1/\lambda) - U : g = 0] = Var[U : g = 0] = 1/\lambda^2$ and thus the contribution of the cycle corresponding to the time interval $(t_{\nu_{2(l-1)}}, t_{\nu_{2l}}]$ towards the variance of $T_{\lambda, F_R, f_B; \nu_{2m}}(0)$ under H_0 is

$$(1/\lambda^2) (\nu_{2l} - \nu_{2l-1}) (1-p)^{\nu_{2l}-\nu_{2(l-1)}-2} p^2.$$

The contribution of all the cycles starting at $t_{\nu_{2(l-1)}}$ and ending at $t_{\nu_{2l}}$ towards this variance is therefore

$$(1/\lambda^2) \sum_{\nu_{2l-1}=\nu_{2(l-1)+1}}^{\nu_{2l}-1} (\nu_{2l} - \nu_{2l-1}) (1-p)^{\nu_{2l}-\nu_{2(l-1)}-2} p^2 = (1/(2\lambda^2)) (\nu_{2l} - \nu_{2(l-1)}) (\nu_{2l} - \nu_{2(l-1)} - 1) (1-p)^{\nu_{2l}-\nu_{2(l-1)}-2} p^2.$$

The overall contribution of a cycle to this variance becomes

$$(p^2/(2\lambda^2)) \sum_{\mu=0}^{\infty} (\mu+1)(\mu+2)(1-p)^{\mu} =$$

$$(p^2/(2\lambda^2)) \partial^2 (\sum_{\mu=0}^{\infty} (1-p)^{\mu+2}) / (\partial p)^2 = 1/(\lambda^2 p).$$

It results

$$Var[T_{\lambda, F_R, f_B; \nu_{2m}}(0) : g = 0] = m/(\lambda^2 p).$$

It follows

$$\sqrt{p/m\lambda} T_{\lambda, F_R, f_B; \nu_{2m}}(0) \sim \Phi$$

under H_0 for m large and the critical region of the asymptotic test of size α is

$$\sqrt{p/m\lambda} t_{\lambda, F_R, f_B; \nu_{2m}}(0) > z(1 - \alpha).$$

5. Should the risk model of Asmussen and Biard be used?

In this model [3] the point process of claim amount arrival instants is a homogeneous Poisson process with global rate λ . A threshold d is introduced for claim interarrival times, such that if the two immediately preceding consecutive both newly observed interarrival times are larger than d the claim size distribution has density f_2 in all other cases the density function is f_1 . At $t = 0$ or just after the appearance of a f_2 -distributed claim a cycle of an embedded renewal process starts and lasts until the end of a sequence of two consecutive new gaps larger than d . Let $(t_{c_1}, t_{c_2}, \dots)$ be the ordered sequence of instances of appearance of a f_2 -distributed claim. Assume that our observation of the aggregated risk process consists of all t_i and the associated claim sizes b_i until the r th cycle of the embedded renewal process ends, i.e. covering the period $(0, t_{c_r}]$. The likelihood function is given by

$$L(\lambda, d, f_1, f_2 : t_1, b_1; \dots; t_{c_r}, b_{c_r}) =$$

$$\prod_{i=1}^{c_r} \lambda e^{-\lambda u_i} \cdot \prod_{k=1}^r f_2(b_{c_k}) \cdot \prod_{j \in \{1, \dots, c_r\} - \{c_1, \dots, c_r\}} f_1(b_j).$$

According to the fundamental theorem of Neyman and Pearson [4] a most powerful test of the classical risk model versus the model of Asmussen and Biard may be based on the test statistic

$$\Lambda =$$

$$\{L(\lambda, d, f_1, f_1 : T(1), B(1); \dots; T(c_r), B(c_r))\}^{-1}.$$

$$L(\lambda, d, f_1, f_2 : T(1), B(1); \dots; T(c_r), B(c_r)) = \prod_{k=1}^r (f_2(B(c_k)) / (f_1(B(c_k)))).$$

An equivalent statistic is

$$\ln(\Lambda) = \sum_{k=1}^r \ln[f_2(B(c_k)) / f_1(B(c_k))].$$

Note that under $H_0 : B(i) \sim f_2 \equiv f_1$ [$i = 1, \dots, c_r$]

$\ln[\Lambda]$ has mean $r(\int_0^\infty \ln[f_2(t)/f_1(t)] f_1(t) dt$ and variance

$$r(\int_0^\infty \ln[f_2(t)/f_1(t)]^2 f_1(t) dt - (\int_0^\infty \ln[f_2(t)/f_1(t)] f_1(t) dt)^2).$$

If these moments are finite and the variance is different from 0 $\ln[\Lambda]$ follows under H_0 a limiting normal distribution and for large r the critical region of the test of size α is specified by

$$[r(\int_0^\infty \ln[f_2(t)/f_1(t)]^2 f_1(t) dt - (\int_0^\infty \ln[f_2(t)/f_1(t)] f_1(t) dt)^2]^{-1/2}$$

$$[\sum_{k=1}^r (\ln[f_2(B(c_k))/f_1(B(c_k))]) - r(\int_0^\infty \ln[f_2(t)/f_1(t)]f_1(t)dt] > z(1 - \alpha).$$

6. Conclusion

As we have seen the recent research achievements in assessing the actuarial risk lead in a natural way to new applications of the methodology of optimal statistical tests for marked point processes.

References

- [1] H. Albrecher and O.J.Boxma, A ruin model with dependence between claim sizes and claim intervals. *Insurance: Mathematics and Economics*, **35** (2004), 245-254.
- [2] M. Boudreault, H. Cossette, D. Landriault, and E. Marceau, On a risk model with dependence between interclaim arrivals and claim sizes. *Scandinavian Actuarial Journal*, **(2006),5**, 265-285.
- [3] S. Asmussen and R.Biard, Ruin probabilities for a regenerative Poisson gap generated risk process. *Eur.Actuar.J.*, **(2011),1**, 3-22.
- [4] D.R. Cox and D. Hinkley, *Theoretical statistics*, Chapman and Hall, London, 1979.

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