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Algebraic Models in Different Fields

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Dedicated to Professor Marius Stoka on the occasion of his 80^{th} *birthday*

Abstract

We consider a preliminary approach to ranking problems, concerning submodels of the Birkhoff model. Being S_n the symmetric group on the data set $\{1, 2, \ldots, n\}$, some results and examples are given for small values of n. [1]

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1 Introduction

Algebraic and geometric models naturally arise in many fields: manufacturing, business, security, engineering. In the last years the Groebner bases theory helped in numerous problems in that areas.The problem of ranking a number of alternatives based on scores or preferences assigned by multiple voters (or under multiple criteria) has become an interesting subject in modern applications. In addition to well-known examples as rankings of colleges, sport teams, stocks, or web pages, ranking methodologies have been used in new specific applications as ranking genes to identify biomarkers, ranking therapeutic chemicals to discover new drugs, ranking manhole disruptions to prioritize maintenance and many other situations. The main discussion is centered

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around the symmetric group S_n on the set of data $\{1, \ldots, n\} = [n]$, of cardinality $n!$. Let $S = K[X_\pi, \pi \in S_n]$ be the polynomial ring with variables indexed by all the permutations π in S_n and whose coefficients are in any infinite field K of characteristic zero (the field of real numbers R). Let $T = K[Y_1, \ldots, Y_n]$ be a polynomial ring, where the indeterminates are viewed as a system of parameters. We call a *Statistical ranking module* a K-algebras homomorphism $f : S \rightarrow T$, and f a toric model, if the images of the variables $f(X_\pi)$ are monomials in the variable Y_i. The classic toric model most known is the so called the *Birkhoff model Bi*. For other models, not necessarily toric, see the interesting paper [5], where some remarks and interesting results are given in view of new applications of the commmutative algebra and algebraic geometry. In this paper, we approach to the generalized situation when S_n is replaced by a proper subset of S_n. In particular, in the N. 1 we recall definitions and results on the Groebner bases theory, crucial for our study, on the classical Birkhoff model. In N. 2 we consider the special case of constraints on the permutations of S_n given by a poset P consisting of one $n-1$ chain and of one incomparable element. For this submodel we compute some algebraic objects connected to it, in particular the dimension of the model polytope. Although the model is poor of results from the mathematical point of view, it is significant from the applied point of view (the incomparable element can represent an element that does not participate to the ranking process, for example a manhole without an order of priority for maintenance), it is the starting point of a systematic study on ranking problems for submodels of the Birkhoff model ([3]). In other cases in fact, there are more interesting open problems.

2 THE BIRKHOFF MODEL Bi

The classical SAMPLING PROBLEM FROM STATISTIC has as ingredients:

1. a group of r voters that are electing a leader
2. a list of n candidates
3. a $n \times n$-matrix A whose entry (i, j) is the number of people who rank the candidate i in the position j.

In fact, each of r voters ranks the n candidates in order of preference, so we obtain a collection of r permutations called permutations data. The matrix A is said the first-order summary or the permutation matrix. The sampling problem consists in choosing at random from the set of all election data (in number of $n!$) with a fixed first-order summary ,in practice, to find all the permutations that produced the given summary matrix. When we translate the problem in a semigroups problem (modelization), it is easy to verify that the set of the $n \times n$-doubly stochastically matrices is a set $\mathbf{A}$ of lattice points, $\mathbf{A} \subset N^{n \times n}$. In other words, each data set is summarized as a function $u : S_n \to N$, where $u(\pi)$ is the number of times the permutation π is observed. Thinking u as a column vector, we can write the matrix Au (matrix-vector product) whose the sum of the entries coincides with the sample size $N =$ sum on the elements $u(\pi)$.

Definition 2.1 *Let $T = K[Y_{ij}, 1 \leq i < j \leq n]$. The Birkhoff model Bi is the semigroups homomorphism $f : S \to T$, such that:$f(X_\pi) = \prod Y_{i,\pi(i)}, 1 \leq i \leq n$, the convex hull $B_n := conv\{\underline{a} \in N^{n \times n} / \underline{Y}^{\underline{a}} \in Imf\}$. $\underline{Y}^{\underline{a}} = Y_{11}^{a_1} \dots Y_{nn}^{a_{n^2}}$ is the Birkhoff polytope $B_n = conv(\mathbf{A})$ and* $\dim(\mathbf{A}) = (n-1)^2 + 1$.

Theorem 2.2 *Let B_n the Birkhoff polytope. Then* $\dim B_n = (n-1)^2$*, being* $\dim B_n$ *the smallest dimension of the affine space containing the polytope.*

Proof: See [4] or [2].

We can define a geometric object: the projective toric variety $Y_{\mathbf{A}}$,in the projective space $P^{n!-1}$, as the closure of the zero locus of the toric ideal $I_{\mathbf{A}}$.

Example 2.3 *For $n = 3, S = K[X_{123}, X_{132}, X_{213}, X_{231}, X_{312}, X_{321}]$. The homomorphism f is such that:*

$$X_{123} \to Y_{11}Y_{22}Y_{33}$$

$$X_{132} \to Y_{11}Y_{23}Y_{32}$$

$$X_{213} \to Y_{12}Y_{21}Y_{33}$$

$$X_{231} \to Y_{12}Y_{23}Y_{31}$$

$$X_{312} \to Y_{13}Y_{21}Y_{32}$$

$$X_{321} \to Y_{13}Y_{22}Y_{31}$$

where (Y_{ij}) is a generic $3 \times 3-$permutation matrix. If $Y_{11} > Y_{12} > Y_{13} > Y_{21} > Y_{22} > Y_{23} > Y_{31} > Y_{32} > Y_{33}$,the set $\mathbf{A}$ of the lattice points are the columns in a doubly-stochastic 9×6-matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and $\dim(B_3) = 4$.

The projective toric variety $Y_{\mathbf{A}}$ *is a hypersuface of degree* 3 *in the projective space* P^5. *Its definition ideal is the toric principal ideal* $I_{\mathbf{A}} = (X_{123}X_{231}X_{312} - X_{132}X_{213}X_{321})$. $I_{\mathbf{A}}$ *is generated by a Groebner basis for any order of the variables and any term order on the monomials of* S, *since* $I_{\mathbf{A}}$ *is principal.*

In the generalized situation when S_n is replaced by a proper subset of S_n, the matrix A has fewer that $n!$ columns, but still labelled by permutations. Let $\mathbf{A}^{'}$ be a subset of lattices points of the model. The resultant model $Bi^{'}$ is a submodel of Bi and for this we require to study:

1. the toric ideal $I_{\mathbf{A}^{'}}$
2. the projective toric variety defined by the toric ideal $I_{\mathbf{A}^{'}}$
3. The Groebner basis of $I_{\mathbf{A}^{'}}$
4. the dimension of the model polytope $Bi^{'}$

For the following, the polynomial ring $S = K[X_\pi, \pi \in S_n]$ will be denoted by $K[\mathbf{A}]$ and for any subset $\mathbf{A}^{'}$ of $\mathbf{A}$, $K[\mathbf{A}^{'}]$ will denote the polynomial ring whose variables are indexed by the elements of $\mathbf{A}^{'}$. We recall two crucial results connected to the Groebner basis theory.

Proposition 2.4 *Let* $\mathbf{A}^{'}$ *be any subset of* $\mathbf{A}$, *then:*

(a) the toric ideal of $\mathbf{A}^{'}$ *is* $I_{\mathbf{A}^{'}} = I_{\mathbf{A}} \bigcap K[\mathbf{A}]$

(b) the Groebner basis $G_{\mathbf{A}^{'}}$ *of* $I_{\mathbf{A}^{'}}$ *is* $G_{\mathbf{A}^{'}} = G_{\mathbf{A}} \bigcap K[\mathbf{A}^{'}]$

Proof (a) obvious. (b) derives from the elimination theory for Groebner bases ([1]).

Theorem 2.5 *Let* $\prec$ *be any of* $(n!)!$ *(graded) reverse-lexicographic order on the variables* X_π. *Then in the ring* $S = K[X_\pi, \pi \in S_n]$, *the initial ideal* $in_{\prec}(I_{\mathbf{A}})$ *is generated by square-free monomials of degree* $\leq n$.

Proof. See [4], Theorem 4.8.

3 Submodels of Bi

We consider proper subsets of all permutations of the set $[n] = \{1, \ldots, n\}$ that are linear extensions $L(P)$ of a given partial order P on $[n]$. The question of describing constraint posets, given by proper subsets $L(P)$ of S_n, in more details is adressed in [5].

Theorem 3.1 *Let P be an arbitrary constraints set on $[n]$. Let $Bi^{'}$ be the restriction to $L(P)$ of the Birkhoff model Bi and $B_n^{'}$ the restriction of its polytope B_n. Consider the sets $Z = \{(i,j) \in [n] \times [n] / \pi(i) \neq j$ for all $\pi \in L(P)\}$ and $C = \{(i,j) \in [n] \times [n]/(i,j) \notin Z$ and $(i,j^{'}) \in Z$ for some $j^{'} > j$ or $(i^{'},j) \in Z$ for some $i^{'} > i\}$. Then $\dim(B_n^{'}) = n^2 - \#Z - \#C$.*

Proof. See [5], Proposition 3.2.

Theorem 3.2 *Let P be the poset given by the $n-1$ chain $1 \prec 2 \prec \ldots \prec n-1$ and one incomparable element. Then we have:*

1) $\#Z = n - 2$

2) $\#C = n^2 - 2n$

Proof. 1)From theorem 3.1, we have to compute the number of couples $(i,j) \in Z, (i,j) \in [n] \times [n]$ such that $\pi(i) \neq j$. We have: $\pi(1) \in \{n,1\}, \pi(2) \in \{n,1,2\}, \pi(3) \in \{n,2,3\}, \ldots, \pi(n-1) \in \{n,n-2,n-1\}, \pi(n) \in \{n,n-1\}$ Then $\#\{(i,j)\} = 2$,for $i = 1,n$ and $\#\{(i,j)\} = 3$, for $i = 2,\ldots,n-1$. Hence $\#Z = 2.2 + (n-2).3 = n - 2$.
2)$(1,j) \in C$, for $j = 2,\ldots,n-1$, $(2,j) \in C$, for $j = 3,\ldots,n-1$, $(3,j) \in C$, for $j = 1,4,\ldots,n-1,\ldots,(n-1,j) \in C$, for $j = 1,2,\ldots,n-3, (n,j) \in C$, for $j = 1,2,\ldots,n-2$. Then $\#C = n^2 - 2n$.

Corollary 3.3 *: Let $B_n^{'}$ be the polytope of the submodel of Bi coming from a poset P consisting of the disjoint union of a $n-1$ chain and 1 incomparable element. Then $\dim B_n^{'} = n + 2$.*

Proof. From the formula contained in Theorem 3.1 and by Theorem 3.2, we obtain $\dim B_n^{'} = n^2 - (n-2) - (n^2 - 2n) = n + 2$.

Remark 3.4 *For $n = 3$, $\dim B_3^{'} = \dim Z\mathbf{A}^{'}$, where $\mathbf{A}^{'} = \{(1,0,0,0,1,0,0,0,1),$ $(1,0,0,0,0,1,0,0,1,0),(0,0,1,0,0,1,0,1,0)\}$.*

Theorem 3.5 *Let P be a poset given by the $n-1$ chain $1 \prec 2 \prec \ldots \prec n-1$ and by one incomparable element. Let $Bi^{'}$ be the submodel of Bi and $I_{\mathbf{A}^{'}}$ the toric ideal of $Bi^{'}$. Then we have:*

1. *$I_{\mathbf{A}^{'}} = (0)$.*

2. *The Groebner basis of $I_{\mathbf{A}^{'}}$ if empty.*

Proof.By theorem 2.5, the Groebner basis of $I_{\mathbf{A}}$ for the degree revlex order has binomial relations of degree $\leq n$ and each initial monomial in the binomial is square free. By theory of elimination, the Groebner basis $G_{\mathbf{A}'}$ of $I_{\mathbf{A}'}$ is of the same type, then the degree of each binomial of $G_{\mathbf{A}'}$ is $\leq n$. We prove that $G_{\mathbf{A}'}$ is empty, hence $I_{\mathbf{A}'} = (0)$. The total number of monomials of the model is n, then, since the monomial $Y_{1n}Y_{21}Y_{32}\dots Y_{nn-1}$ is unique, the variable $T_{n12\dots n-1}$ does not appear in a binomial relation, then we have at maximum $n-1$ variables involved in the binomial relations. This depends from the fact that the variable Y_{1n} appears only in one monomial. Similarly, the variable Y_{2n} appears only in one monomial and so on until $Y_{n-1,n}$. It follows that we cannot have binomials of any degree $\leq n$, that are binomial relations.

It is important to investigate for other submodels. The first derives from a poset P, a disjoint union of 2 chains $1 \prec 2 \prec \dots \prec n/2$ and $n/2+1 \prec n/2+2 \prec \dots \prec n$ of $n/2$ elements, if n is even.
We conclude with a small example, that describes a submodel of Bi , for $n=4$ and poset P a disjoint union of two chains $1 \prec 2$ and $3 \prec 4$. We obtain the toric ideal $I_{\mathbf{A}'}$ by simple statements that can be followed in the general case, for reducing the number of variables involved in the binomial generators. It is obvious that, being n small, we can utilize the software Macaulay2 to find $I_{\mathbf{A}}$ and $I_{\mathbf{A}'}$.

Proposition 3.6 *For $n=4$ and constraints given by a poset P consisting of two chains $1 \prec 2$ and $3 \prec 4$, we find:*

1. $I_{\mathbf{A}'} = (T_{1324}T_{3142} - T_{1342}T_{3124})$
2. $\dim B_4' = 5$

Proof : We have six permutations $1234, 1342, 3142, 1342, 3124, 3412$.
1. By Theorem 3.2, the initial ideal of $I_{\mathbf{A}'}$, for the degree reverse lexicographic order, is square-free and generated in degree ≤ 4. Since the variable Y_{22} appears only in the monomial $Y_{11}Y_{22}Y_{33}Y_{44}$ of $K[Y_{11},\dots,Y_{44}]$, the variable T_{1234} cannot appear in a binomial generator of $I_{\mathbf{A}'}$. The variable Y_{31} appears only in the monomial $Y_{13}Y_{24}Y_{31}Y_{42}$,then the variable T_{1234} cannot appear in a binomial generator of $I_{\mathbf{A}'}$ .Then we have only 4 variables $T_{1324}, T_{1342}, T_{3142}, T_{3124}$ involved in generator binomial relations and we cannot have generators of degree 4. Concerning the degree 3, none product of three variables can appear as a square-free initial term of a binomial generator of $I_{\mathbf{A}'}$,because the(not necessary square-free) monomial that completes the binomial must be a thirty power of the remaining variable, absurd. Then, the only binomial relation is $T_{1324}T_{3142} - T_{1342}T_{3124}$.

2. Using theorem 3.4, we have to compute the cardinality of the sets Z and C.
In this case, $Z = \{(1,2),(1,4),(4,1),(4,3)\}, C = \{(1,1),(1,3),(2,1),(2,3),(3,1),(3,3),(4,2)\}$.
Then $\dim B_4^{'} = 16 - 4 - 7 = 5$.

From the previus proposition, we can deduce some informations in the general case, about the variables involved in a binomial generator of the toric ideal.

Proposition 3.7 *For constraints given by a poset P consisting ot two chains $1 \prec 2 \ldots \prec n/2$ and $n/2{+}1 \ldots n$, the variables $T_{1,2\ldots,n/2\ldots,n}$ and $T_{n/2+1,n/2+2,\ldots,n,1,2\ldots,n/2}$ cannot appear in a binomial generator of $I_{\mathbf{A}'}$.*

Proof We have $\pi(n/2) = n/2$ only in the identic permutation and $\pi(n/2+1)$ only in the permutation $n/2+1 \;\; n/2+2 \;\; \ldots n \;\; 1 \;\; 2 \ldots \; n/2$. Then the variables $Y_{n/2,n/2}$ and$Y_{n/2+1,1}$ appear only in a square-free monomial of degree n in the variables Y_{ij}. As a consequence the variables $T_{1,2\ldots n/2,n/2+1,\ldots,n}$ and $T_{n/2+1,n/2+2,\ldots,n,1,2,\ldots,n/2}$ are not involved in binomial generators of $I_{\mathbf{A}'}$.

Theorem 3.2 can be improved to obtain better upper bounds for submodules $Bi^{'}$ obtained by constraints given by posets.

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