Algebraic Models in Different Fields

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Dedicated to Professor Marius Stoka on the occasion of his 80th birthday

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Abstract

We consider a preliminary approach to ranking problems, concerning submodels of the Birkhoff model. Being $S_n$ the symmetric group on the data set $\{1, 2, \ldots, n\}$, some results and examples are given for small values of $n$.  

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1 Introduction

Algebraic and geometric models naturally arise in many fields: manufacturing, business, security, engineering. In the last years the Groebner bases theory helped in numerous problems in that areas. The problem of ranking a number of alternatives based on scores or preferences assigned by multiple voters (or under multiple criteria) has become an interesting subject in modern applications. In addition to well-known examples as rankings of colleges, sport teams, stocks, or web pages, ranking methodologies have been used in new specific applications as ranking genes to identify biomarkers, ranking therapeutic chemicals to discover new drugs, ranking manhole disruptions to prioritize maintenance and many other situations. The main discussion is centered

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around the symmetric group $S_n$ on the set of data $\{1, \ldots, n\} = [n]$, of cardinality $n!$. Let $S = K[X_{\pi}, \pi \in S_n]$ be the polynomial ring with variables indexed by all the permutations $\pi$ in $S_n$ and whose coefficients are in any infinite field $K$ of characteristic zero (the field of real numbers $R$). Let $T = K[Y_1, \ldots, Y_n]$ be a polynomial ring, where the indeterminates are viewed as a system of parameters. We call a Statistical ranking module a $K$-algebras homomorphism $f : S \rightarrow T$, and $f$ a toric model, if the images of the variables $f(X_{\pi})$ are monomials in the variable $Y_i$. The classic toric model most known is the so-called the Birkhoff model $Bi$. For other models, not necessarily toric, see the interesting paper [5], where some remarks and interesting results are given in view of new applications of the commutative algebra and algebraic geometry.

In this paper, we approach to the generalized situation when $S_n$ is replaced by a proper subset of $S_n$. In particular, in the N. 1 we recall definitions and results on the Groebner bases theory, crucial for our study, on the classical Birkhoff model. In N. 2 we consider the special case of constraints on the permutations of $S_n$ given by a poset $P$ consisting of one $n - 1$ chain and of one incomparable element. For this submodel we compute some algebraic objects connected to it, in particular the dimension of the model polytope. Although the model is poor of results from the mathematical point of view, it is significant from the applied point of view (the incomparable element can represent an element that does not participate to the ranking process, for example a manhole without an order of priority for maintenance), it is the starting point of a systematic study on ranking problems for submodels of the Birkhoff model ([3]). In other cases in fact, there are more interesting open problems.

2 THE BIRKHOFF MODEL $Bi$

The classical SAMPLING PROBLEM FROM STATISTIC has as ingredients:

1. a group of $r$ voters that are electing a leader
2. a list of $n$ candidates
3. a $n \times n$-matrix $A$ whose entry $(i, j)$ is the number of people who rank the candidate $i$ in the position $j$.

In fact, each of $r$ voters ranks the $n$ candidates in order of preference, so we obtain a collection of $r$ permutations called permutations data. The matrix $A$ is said the first-order summary or the permutation matrix. The sampling problem consists in choosing at random from the set of all election data (in number of $n!$) with a fixed first-order summary, in practice, to find all the permutations that produced the given summary matrix. When we translate the problem in a semigroups problem (modelization), it is easy to
verify that the set of the $n \times n$-doubly stochastically matrices is a set $A$ of lattice points, $A \subset N^{n \times n}$. In other words, each data set is summarized as a function $u : S_n \to N$, where $u(\pi)$ is the number of times the permutation $\pi$ is observed. Thinking $u$ as a column vector, we can write the matrix $Au$ (matrix-vector product) whose the sum of the entries coincides with the sample size $N = \text{sum on the elements } u(\pi)$.

**Definition 2.1** Let $T = K[Y_{ij}, 1 \leq i < j \leq n]$. The Birkhoff model $B_i$ is the semigroups homomorphism $f : S \to T$, such that: $f(X_\pi) = \prod Y_{i,\pi(i)}, 1 \leq i \leq n,$ the convex hull $B_n := \text{conv}\{a \in N^{n \times n} \mid Y^n a \in \text{Im } f\}$. $Y^n a = Y_{11}^a \ldots Y_{nn}^a$ is the Birkhoff polytope $B_n = \text{conv}(A)$ and $\text{dim}(A) = (n - 1)^2 + 1$.

**Theorem 2.2** Let $B_n$ the Birkhoff polytope. Then $\text{dim } B_n = (n - 1)^2$, being $\text{dim } B_n$ the smallest dimension of the affine space containing the polytope.

**Proof:** See [4] or [2].

We can define a geometric object: the projective toric variety $Y_A$, in the projective space $P^{n! - 1}$, as the closure of the zero locus of the toric ideal $I_A$.

**Example 2.3** For $n = 3$, $S = K[X_{123}, X_{132}, X_{213}, X_{231}, X_{312}, X_{321}]$. The homomorphism $f$ is such that:

\[
\begin{align*}
X_{123} &\to Y_{11} Y_{22} Y_{33} \\
X_{132} &\to Y_{11} Y_{23} Y_{32} \\
X_{213} &\to Y_{12} Y_{21} Y_{33} \\
X_{231} &\to Y_{12} Y_{23} Y_{31} \\
X_{312} &\to Y_{13} Y_{21} Y_{32} \\
X_{321} &\to Y_{13} Y_{22} Y_{31}
\end{align*}
\]

where $(Y_{ij})$ is a generic $3 \times 3$-permutation matrix. If $Y_{11} > Y_{12} > Y_{13} > Y_{21} > Y_{22} > Y_{23} > Y_{31} > Y_{32} > Y_{33}$, the set $A$ of the lattice points are the columns in a doubly-stochastic $9 \times 6$-matrix

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]
and $\dim(B_3) = 4$.

The projective toric variety $Y_A$ is a hypersurface of degree 3 in the projective space $\mathbb{P}^5$. Its definition ideal is the toric principal ideal $I_A = (X_{123}X_{231}X_{312} - X_{132}X_{213}X_{321})$. $I_A$ is generated by a Groebner basis for any order of the variables and any term order on the monomials of $S$, since $I_A$ is principal.

In the generalized situation when $S_n$ is replaced by a proper subset of $S_n$, the matrix $A$ has fewer than $n!$ columns, but still labelled by permutations. Let $A'$ be a subset of lattice points of the model. The resultant model $Bi'$ is a submodel of $Bi$ and for this we require to study:

1. the toric ideal $I_{A'}$
2. the projective toric variety defined by the toric ideal $I_{A'}$
3. The Groebner basis of $I_{A'}$
4. the dimension of the model polytope $Bi'$

For the following, the polynomial ring $S = K[X_\pi, \pi \in S_n]$ will be denoted by $K[A]$ and for any subset $A'$ of $A$, $K[A']$ will denote the polynomial ring whose variables are indexed by the elements of $A'$. We recall two crucial results connected to the Groebner basis theory.

**Proposition 2.4** Let $A'$ be any subset of $A$, then:

(a) the toric ideal of $A'$ is $I_{A'} = I_A \cap K[A]$

(b) the Groebner basis $G_{A'}$ of $I_{A'}$ is $G_{A'} = G_A \cap K[A']$

**Proof** (a) obvious. (b) derives from the elimination theory for Groebner bases ([1]).

**Theorem 2.5** Let $\prec$ be any of $(n!)!$ (graded) reverse-lexicographic order on the variables $X_\pi$. Then in the ring $S = K[X_\pi, \pi \in S_n]$, the initial ideal $\text{in}_{\prec}(I_A)$ is generated by square-free monomials of degree $\leq n$.

**Proof.** See [4], Theorem 4.8.

### 3 Submodels of $Bi$

We consider proper subsets of all permutations of the set $[n] = \{1, \ldots, n\}$ that are linear extensions $L(P)$ of a given partial order $P$ on $[n]$. The question of describing constraint posets, given by proper subsets $L(P)$ of $S_n$, in more details is addressed in [5].
Theorem 3.1 Let $P$ be an arbitrary constraints set on $[n]$. Let $B_i'$ be the restriction to $L(P)$ of the Birkhoff model $B_i$ and $B_n'$ the restriction of its polytope $B_n$. Consider the sets $Z = \{(i,j) \in [n] \times [n]/\pi(i) \neq j \text{ for all } \pi \in L(P)\}$ and $C = \{(i,j) \in [n] \times [n]/(i,j) \notin Z \text{ and } (i',j') \in Z \text{ for some } j' > j \text{ or } (i',j) \in Z \text{ for some } i' > i\}$. Then $\dim(B_n') = n^2 - \#Z - \#C$.

**Proof.** See [5], Proposition 3.2.

**Theorem 3.2** Let $P$ be the poset given by the $n-1$ chain $1 \prec 2 \prec \ldots \prec n-1$ and one incomparable element. Then we have:

1) $\# Z = n - 2$

2) $\# C = n^2 - 2n$

**Proof.** 1) From Theorem 3.1, we have to compute the number of couples $(i,j) \in Z, (i,j) \in [n] \times [n]$ such that $\pi(i) \neq j$. We have: $\pi(1) \in \{n,1\}, \pi(2) \in \{n,1,2\}, \pi(3) \in \{n,2,3\}, \ldots, \pi(n-1) \in \{n,n-2,n-1\}, \pi(n) \in \{n,n-1\}$

Then $\# \{(i,j)\} = 2$, for $i = 1, n$ and $\# \{(i,j)\} = 3$, for $i = 2, \ldots, n-1$. Hence $\# Z = 2 + (n-2) \cdot 3 = n - 2$.

2) $(1,j) \in C$, for $j = 2, \ldots, n-1$, $(2,j) \in C$, for $j = 3, \ldots, n-1$, $(3,j) \in C$, for $j = 1, 4, \ldots, n-1, \ldots, (n-1,j) \in C$, for $j = 1, 2, \ldots, n-3, (n,j) \in C$, for $j = 1, 2, \ldots, n-2$. Then $\# C = n^2 - 2n$.

**Corollary 3.3** : Let $B_n'$ be the polytope of the submodel of $B_i$ coming from a poset $P$ consisting of the disjoint union of a $n-1$ chain and 1 incomparable element. Then $\dim B_n' = n + 2$.

**Proof.** From the formula contained in Theorem 3.1 and by Theorem 3.2, we obtain $\dim B_n' = n^2 - (n-2) - (n^2 - 2n) = n + 2$.

**Remark 3.4** For $n = 3$, $\dim B_3' = \dim Z \mathbf{A}'$, where $\mathbf{A}' = \{(1,0,0,0,1,0,0,0,1), (1,0,0,0,0,1,0,0,1,0), (0,0,1,0,0,1,0,1,0)\}$.

**Theorem 3.5** Let $P$ be a poset given by the $n-1$ chain $1 \prec 2 \prec \ldots \prec n-1$ and by one incomparable element. Let $B_i'$ be the submodel of $B_i$ and $I_{\mathbf{A}'}$ the toric ideal of $B_i'$. Then we have:

1. $I_{\mathbf{A}'} = (0)$.

2. The Groebner basis of $I_{\mathbf{A}'}$ if empty.
Proof: By theorem 2.5, the Groebner basis of $I_A$ for the degree revlex order has binomial relations of degree $\leq n$ and each initial monomial in the binomial is square free. By theory of elimination, the Groebner basis $G_{A'}$ of $I_{A'}$ is of the same type, then the degree of each binomial of $G_{A'}$ is $\leq n$. We prove that $G_{A'}$ is empty, hence $I_{A'} = (0)$. The total number of monomials of the model is $n$, then, since the monomial $Y_{1n}Y_{22}Y_{33}Y_{44}$ of $K[Y_{11},\ldots,Y_{44}]$, the variable $T_{1324}$ cannot appear in a binomial relation, then we have at maximum $n - 1$ variables involved in the binomial relations. This depends from the fact that the variable $Y_{1n}$ appears only in one monomial. Similarly, the variable $Y_{2n}$ appears only in one monomial and so on until $Y_{n-1,n}$. It follows that we cannot have binomials of any degree $\leq n$, that are binomial relations.

It is important to investigate for other submodels. The first derives from a poset $P$, a disjoint union of 2 chains $1 \prec 2 \prec \ldots \prec n/2$ and $n/2 + 1 \prec \ldots \prec n$ of $n/2$ elements, if $n$ is even. We conclude with a small example, that describes a submodel of $Bi$, for $n = 4$ and poset $P$ a disjoint union of two chains $1 \prec 2$ and $3 \prec 4$. We obtain the toric ideal $I_{A'}$ by simple statements that can be followed in the general case, for reducing the number of variables involved in the binomial generators. It is obvious that, being $n$ small, we can utilize the software Macaulay2 to find $I_A$ and $I_{A'}$.

Proposition 3.6 For $n = 4$ and constraints given by a poset $P$ consisting of two chains $1 \prec 2$ and $3 \prec 4$, we find:

1. $I_{A'} = (T_{1324}T_{3142} - T_{1342}T_{3124})$
2. $\dim B'_4 = 5$

Proof: We have six permutations $1234, 1342, 3142, 1342, 3124, 3412$.

1. By Theorem 3.2, the initial ideal of $I_{A'}$, for the degree reverse lexicographic order, is square-free and generated in degree $\leq 4$. Since the variable $Y_{22}$ appears only in the monomial $Y_{11}Y_{22}Y_{33}Y_{44}$ of $K[Y_{11},\ldots,Y_{44}]$, the variable $T_{1324}$ cannot appear in a binomial generator of $I_{A'}$. The variable $Y_{31}$ appears only in the monomial $Y_{13}Y_{24}Y_{31}Y_{42}$, then the variable $T_{1234}$ cannot appear in a binomial generator of $I_{A'}$. Then we have only 4 variables $T_{1324}, T_{1342}, T_{3142}, T_{3124}$ involved in generator binomial relations and we cannot have generators of degree 4. Concerning the degree 3, none product of three variables can appear as a square-free initial term of a binomial generator of $I_{A'}$, because the (not necessary square-free) monomial that completes the binomial must be a thirty power of the remaining variable, absurd. Then, the only binomial relation is $T_{1324}T_{3142} - T_{1342}T_{3124}$. 
2. Using theorem 3.4, we have to compute the cardinality of the sets \( Z \) and \( C \).
In this case, \( Z = \{(1, 2), (1, 4), (4, 1), (4, 3)\} \), \( C = \{(1, 1), (1, 3), (2, 1), (2, 3), (3, 1), (3, 3), (4, 2)\} \).
Then \( \dim B'_4 = 16 - 4 - 7 = 5 \).

From the previous proposition, we can deduce some informations in the general case, about the variables involved in a binomial generator of the toric ideal.

**Proposition 3.7** For constraints given by a poset \( P \) consisting of two chains \( 1 \prec 2 \ldots \prec n/2 \) and \( n/2+1 \ldots n \), the variables \( T_{1,2,n/2+1} \ldots n \) and \( T_{n/2+1,n/2+2} \ldots n,1,2 \ldots n/2 \) cannot appear in a binomial generator of \( I_\mathbf{A} \).

**Proof** We have \( \pi(n/2) = n/2 \) only in the identric permutation and \( \pi(n/2 + 1) \) only in the permutation \( n/2 + 1 \ n/2 + 2 \ldots n \ 1 \ 2 \ldots n/2 \). Then the variables \( Y_{n/2,n/2} \) and \( Y_{n/2+1,1} \) appear only in a square-free monomial of degree \( n \) in the variables \( Y_{ij} \). As a consequence the variables \( T_{1,2,n/2,n/2+1} \ldots n \) and \( T_{n/2+1,n/2+2} \ldots n,1,2 \ldots n/2 \) are not involved in binomial generators of \( I_\mathbf{A} \).

Theorem 3.2 can be improved to obtain better upper bounds for submodules \( B_t \) obtained by constraints given by posets.

**References**


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