Lower and Upper Bounds for Chord Power Integrals of Ellipsoids

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Dedicated to Professor Marius Stoka on the occasion of his 80th birthday

Abstract

First we discuss different representations of chord power integrals $I_p(K)$ of any order $p \geq 0$ for convex bodies $K \subset \mathbb{R}^d$ with inner points. Second we derive closed-term expressions of $I_p(E(a))$ for an ellipsoid $E(a)$ with semi-axes $a = (a_1, \ldots, a_d)$ in terms of the support function of $E(a)$ and prove upper and lower bounds expressed by the volume and the mean breadth of $E(a)$, respectively. A further inequality conjectured in Davy (1984) is proved for ellipsoids. Some remarks on chord power integrals of superellipsoids and simplices round off the topic.

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1 Chord Power Integrals - Definition and Basics

Let $K$ be a convex body in $\mathbb{R}^d$ with interior points and $\mathbb{S}^{d-1} = \partial \mathbb{B}^d$ the boundary of the Euclidean unit ball $\mathbb{B}^d = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$. Further, let $\mathcal{H}^k$ denote
the $k$–dimensional Hausdorff measure on $\mathbb{R}^d$ for $k = 1, \ldots, d$. As usual let us denote by $V(K) = \mathcal{H}^d(K)$ and $S(K) = \mathcal{H}^{d-1}(\partial K)$ volume and surface content of $K$, respectively. Further, we recall that $\kappa_d := V(\mathbb{B}^d) = \pi^{d/2}/\Gamma(d/2 + 1)$ and $S(\mathbb{B}^d) = d \kappa_d$ with $\Gamma(s) := \int_0^\infty e^x x^{s-1}dx$ for $s > 0$.

For any $p \geq 0$ we define the $p$th-order chord power integral (CPI) of $K$ by

$$\mathcal{I}_p(K) = \frac{1}{2} \int_{S^{d-1}} \int_{\mathbb{B}^d} \big( \mathcal{H}^{1}(K \cap \ell(x,u)) \big)^p d\mathcal{H}^{d-1}(du)$$

(with $0^0 := 0$), where $\ell(x,u) := \{x + \alpha u : \alpha \in \mathbb{R}\}$ stands for the line in direction $u \in S^{d-1}$ through $x \in \mathbb{R}^d$ and $K|u^\perp$ is the orthogonal projection of $K$ on $u^\perp$ (= $(d-1)$-dimensional subspace orthogonal to $u$). CPI’s are of considerable interest in integral and stochastic geometry for a long time, see [11], [13], and have many applications in material sciences, physics and image analysis, see e.g. [4], [5], [14] and references therein. In textbooks of integral and convex geometry, see e.g. [11], [13], the r.h.s. of (1) is mostly written as integral w.r.t. the line measure $\mu_1^{(d)}(\cdot)$ (defined on the space $\mathbb{A}(d,1)$ of one-dimensional affine subspaces of $\mathbb{R}^d$):

$$\mathcal{I}_p(K) = \frac{d \kappa_d}{2} \int_{\mathbb{A}(d,1)} \big( \mathcal{H}^{1}(K \cap L) \big)^p \mu_1^{(d)}(dL),$$

where, for integers $p = 2, \ldots, d$, the Blaschke-Petkantschin formula, see [11] (p. 363) provides the representations

$$\mathcal{I}_{k+1}(K) = \frac{(k+1) d \kappa_d}{2 \kappa_k} \int_{\mathbb{A}(d,k)} \big( \mathcal{H}^{k}(K \cap L) \big)^2 \mu_k^{(d)}(dL)$$

for $k = 1, \ldots, d$ with the motion-invariant $k$-flat measure $\mu_k^{(d)}(\cdot)$ (defined on the space $\mathbb{A}(d, k)$ of $k$-dimensional affine subspaces of $\mathbb{R}^d$) satisfying the normalization $\mu_k^{(d)}(\{E \in \mathbb{A}(d, k) : E \cap \mathbb{B}^d \neq \emptyset\}) = \kappa_{d-k}$. From (1) for $p = 0, 1$ and (3) for $k = d - 1$ we get the following relations, see e.g. [12],

$$\mathcal{I}_0(K) = \frac{\kappa_{d-1}}{2} S(K), \quad \mathcal{I}_1(K) = \frac{d \kappa_d}{2} V(K), \quad \mathcal{I}_{d+1}(K) = \frac{d(d+1)}{2} V(K)^2.$$

**Remark 1.** The moments $m_k(K) := \mathcal{I}_k(K)/\mathcal{I}_0(K) = E L_{\mu,K}^k$ for $k = 1, 2, \ldots$ determine a unique distribution function $F_{\mu,K}$ (of the length $L_{\mu,K}$ of the $\mu$-random chord of $K$) which, however, does not characterize the shape of $K$ completely, see [11].

Due to F. Piefke [9], see also [12], the r.h.s. of (1) can be expressed for any $p > 1$ by the distribution of the interpoint distance of two randomly chosen
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points inside $K$ leading to

$$\frac{2I_p(K)}{p(p-1)} = \int_K \int_K \frac{dx\,dy}{\|x-y\|^{d+1-p}} = \int_{K \oplus (-K)} \frac{V(K \cap (K + x))}{\|x\|^{d+1-p}}$$

for any real $p > 1$, i.e., the ratio $I_p(K)/V(K)^2$ takes the form

$$\frac{I_p(K)}{V(K)^2} = \frac{p(p-1)}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{U_K(dx)U_K(dy)}{\|x-y\|^{d+1-p}} \quad \text{with} \quad U_K(\cdot) = \frac{\mathcal{H}^d(\cdot \cap K)}{\mathcal{H}^d(K)}$$

and the integral on r.h.s of the last line is known as $(d+1-p)$-energy of the probability measure $U_K$ (= uniform distribution on $K$).

2 CLT for a Class Poisson Cylinder Processes

To motivate our study of CPI’s we state a central limit theorem (CLT) for the total volume of the union of isotropic Poisson $k$-cylinders included in an expanding convex domain $\varrho K$ as $\varrho \uparrow \infty$, where $K \subset \mathbb{R}^d$ is a convex body containing the origin $o$ of $\mathbb{R}^d$ as inner point. To be precise we need some further notation. For details and the proof of the below CLT the reader is referred to [7].

Let $\Pi_\lambda = \{P_i\}_{i \geq 1}$ be a stationary Poisson point process on $\mathbb{R}^{d-k}$ with positive intensity $\lambda := \mathbb{E}\{i \geq 1 : P_i \in [0,1]^{d-k}\}$, where each point $P_i$ is associated with an independent copy $(\Xi_i, \Theta_i)$ of some generic pair $(\Xi_0, \Theta_0)$ which consists of a random compact set $\Xi_0 \subset \mathbb{R}^{d-k}$ satisfying $0 < \mathcal{H}^{d-k}(\Xi_0)^2 < \infty$ and a random orthogonal matrix $\Theta_0$ whose (uniform) distribution is induced by the normalized Haar measure on the Grassmannian of $k$-dimensional subspaces in $\mathbb{R}^d$. In addition, the components $\Xi_0$ and $\Theta_0$ are independent and the sequence $\{(\Xi_i, \Theta_i)\}_{i \geq 1}$ is generated stochastically independent of $\Pi_\lambda$.

The countable family of random closed sets in $\mathbb{R}^d$

$$\Theta_i((\Xi_i \oplus P_i) \times \mathbb{R}^k) := \{\Theta_i(\xi + P_i, x) \in \mathbb{R}^d : \xi \in \Xi_i, x \in \mathbb{R}^k\} \quad \text{for} \quad i \geq 1$$

forms a stationary and isotropic process of Poisson $k$-cylinders in $\mathbb{R}^d$ for $k = 1, \ldots, d-1$. The motion-invariant union set of the Poisson $k$-cylinders

$$\Xi_{d,k} := \bigcup_{i \geq 1} \Theta_i((\Xi_i \oplus P_i) \times \mathbb{R}^k)$$

for each $k = 1, \ldots, d-1$

is observed in an unboundedly increasing convex window $\varrho K$ as $\varrho \uparrow \infty$, see Fig. 1 and Fig. 2 for realizations of $\Xi_{2,1}$ and $\Xi_{3,1}$ with $K = [0,1]^2$ and $K = [0,1]^3$, respectively.
Now we are in a position to formulate the announced CLT for the volume fraction of $\Xi_{d,k}$ in $\varrho K$ for $k = 1, \ldots, d - 1$: As $\varrho \uparrow \infty$, the random variable

$$\varrho^{(d-k)/2} \left( \frac{\mathcal{H}^d(\Xi_{d,k} \cap \varrho K)}{\varrho^d V(K)} - \mathbb{E}\mathcal{H}^d(\Xi_{d,k} \cap [0,1]^d) \right)$$

is asymptotically normally distributed with mean zero and variance

$$\sigma^2(K) = \frac{2 \kappa_k \lambda M_2 e^{-2\lambda M_1}}{(k + 1) d \kappa_d} \frac{I_{k+1}(K)}{V(K)^2},$$

where $\mathbb{E}\mathcal{H}^d(\Xi_{d,k} \cap [0,1]^d) = 1 - e^{-\lambda M_1}$ with $M_j := \mathbb{E}\mathcal{H}^{d-k}(\Xi_0)^j$, $j = 1, 2$.

This CLT is of interest from several points of view. The Gaussian limit of (5) depends on the shape of the observation window, i.e. on $K$, expressed in terms of the CPI $I_{k+1}(K)$. This shape-dependence of $\sigma^2(K)$ is caused by the intrinsic long-range correlations of the random set $\Xi_{d,k}$. In contrast to this, in case of random sets satisfying certain weak dependence conditions, e.g. as in the degenerate case $k = 0$, the asymptotic variance $\sigma^2(K)$ depends only on the volume $V(K)$. Note that $\Xi_{d,0}$ can be identified with a stationary Boolean model with typical grain $\Xi_0$, see [13] (p. 117).

Statisticians aim at designing observation procedures such that estimators of model characteristics have minimal variances. In our model this means to minimize the ratio $I_{k+1}(K)/V(K)^2$ in (6) if another ovoid functional of $K$, e.g. the mean breadth $b(K)$, is fixed. In the planar case optimal lower bounds of $\mathcal{I}_2(K)/\mathcal{H}^2(K)^2$ have been obtained for particular classes of convex sets in [6] when the perimeter $\mathcal{H}^1(\partial K)$ is given. In convex geometry, see [3] or [13],
one is also interested to maximize $\mathcal{I}_{k+1}(K)$ when $V(K)$ is fixed. Among all convex bodies the ball with radius $(V(K)/\kappa_d)^{1/d}$ is the unique maximizer due to Carleman’s inequality

$$\mathcal{I}_{k+1}(K) \leq \frac{2^k d \kappa_d \kappa_{d+k}}{\kappa_{k+1}} \left( \frac{V(K)}{\kappa_d} \right)^{(d+k)/d} \quad \text{for } k = 0, 1, \ldots, d.$$  \hspace{1cm} (7)

In the main part of the paper we study CPI’s of $d$-dimensional ellipsoids $\mathbb{E}(a)$ with positive semi-axes $a = (a_1, \ldots, a_d)$ defined by

$$\mathbb{E}(a) = \{x = (x_1, \ldots, x_d) : x_1^2/a_1^2 + \cdots + x_d^2/a_d^2 \leq 1\}. \hspace{1cm} (8)$$

### 3 A Formula for CPI’s of Ellipsoids

From (8) it is easily seen that the diagonal matrix $A = \text{diag}[a_1, \ldots, a_d]$ maps the unit ball $\mathbb{E}^d$ onto $\mathbb{E}(a)$, i.e. $\mathbb{E}(a) = A \mathbb{E}^d$, which implies the well-known formula $V(\mathbb{E}(a)) = \kappa_d \det(A) = \kappa_d a_1 \cdots a_d$. The other intrinsic volumes $V_j(K)$, $j = 1, \ldots, d-1$, of the convex body $K$, see [13] (p. 600), with twice continuously differentiable boundary $\partial K$ can be expressed by the integral $\int_{S^{d-1}} D_j(H_K(u)) \mathcal{H}^{d-1}(du)/(d-j) \kappa_{d-j}$, see [1]. In the latter formula the integrand is the sum of the $(d-j)$ principal $j$-minors of the Hessian matrix $H_K(u) := (\partial^2 h_K(u)/\partial u_i \partial u_j)_{i,j=1}^d$ of the support function $h_K(u) := \max\{\langle u, x \rangle : x \in K\}$ of $K$ for $u = (u_1, \ldots, u_d) \in S^{d-1}$, where $\langle u, x \rangle = u_1 x_1 + \cdots + u_d x_d$.

Basic facts on support functions and their role in convex geometry can be found in [1] and [13]. For example, the mean breadth $b(K) = 2 \kappa_{d-1} V_1(K)/d \kappa_d$ and $S(K) = 2 V_{d-1}(K)$ can be written in terms of $h_K(\cdot)$ as follows:

$$b(K) = \frac{2}{d \kappa_d} \int_{S^{d-1}} h_K(u) \mathcal{H}^{d-1}(du), \quad S(K) = \int_{S^{d-1}} D_{d-1}(H_K(u)) \mathcal{H}^{d-1}(du).$$

The support function of the ellipsoid $\mathbb{E}(a)$ defined in (8) is well-known and $D_{d-1}(H_{\mathbb{E}(a)}(u))$ can be expressed (after rather lengthy calculations) by some power of $h_{\mathbb{E}(a)}(u)$ for $u \in S^{d-1}$, more precisely,

$$h_{\mathbb{E}(a)}(u) = \sqrt{a_1^2 u_1^2 + \cdots + a_d^2 u_d^2} \quad \text{and} \quad D_{d-1}(H_{\mathbb{E}(a)}(u)) = \frac{a_1^2 \cdots a_d^2}{h_{\mathbb{E}(a)}(u)^{d+1}}. \hspace{1cm} (9)$$

This yields an integral expression of $\mathcal{I}_0(\mathbb{E}(a)) = \kappa_{d-1} S(\mathbb{E}(a))/2$. The remaining CPI’s of $\mathbb{E}(a)$ are given in
Theorem 3.1 For any real \( p \geq 0 \), we have

\[
\frac{\mathcal{I}_p(\mathbb{E}(a))}{V^2(\mathbb{E}(a))} = \frac{2^{p-1} \Gamma^2 \left( \frac{d}{2} + 1 \right) \Gamma \left( \frac{p}{2} + 1 \right)}{\pi^{(d+1)/2} \Gamma \left( \frac{d+p+1}{2} \right)} \int_{S^{d-1}} \frac{\mathcal{H}^{d-1}(du)}{h_{\mathbb{E}(a)}(u)^{d+1-p}}. \tag{10}
\]

Proof. To begin with we rewrite (4) with indicator functions w.r.t. the ellipsoid \( K = \mathbb{E}(a) \) so that, for any \( p > 1 \),

\[
\frac{2 \mathcal{I}_p(\mathbb{E}(a))}{p(p-1)} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1_{\mathbb{E}(a)}(x)1_{\mathbb{E}(a)}(y)}{\|x-y\|^{d+1-p}} \, dx \, dy. \tag{11}
\]

By substituting \( x = A s \) and \( y = A t \) for \( s, t \in \mathbb{B}^d \) and applying the integral transformation formula twice the double integral on the r.h.s. of (11) takes the form

\[
\prod_{i=1}^d \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1_{\mathbb{B}^d}(s)1_{\mathbb{B}^d}(t)}{\|A(s-t)\|^{d+1-p}} \, ds \, dt = \frac{V(\mathbb{E}(a))^2}{\kappa_d^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{V(\mathbb{B}^d \cap (\mathbb{B}^d + z))\, dz}{\|Az\|^{d+1-p}}.
\]

Next, we introduce spherical coordinates \( z = r u \) for \( r = \|z\| \geq 0 \) and \( u = (u_1, \ldots, u_d) \in S^{d-1} \). Since the covariogram \( V(\mathbb{B}^d \cap (\mathbb{B}^d + ru)) = V(\mathbb{B}^d \cap (\mathbb{B}^d + r,0,\ldots,0)) =: g_{\mathbb{B}^d}(r) \) of the unit ball depends only on \( r \geq 0 \) and disappears for \( r > 2 \) we find together with the infinitesimal transformation rule \( dz = r^{d-1} dr \, \mathcal{H}^{d-1}(du) \) and \( \|Az\| = r \|Au\| = r h_{\mathbb{E}(a)}(u) \) that

\[
\frac{\mathcal{I}_p(\mathbb{E}(a))}{V(\mathbb{E}(a))^2} = \frac{p(p-1)}{2 \kappa_d^2} \int_0^2 \int_{S^{d-1}} g_{\mathbb{B}^d}(r) r^{d-1} \, \mathcal{H}^{d-1}(du) \, dr \frac{\|rAu\|^{d+1-p}}{\|Au\|^{d+1-p}} = \frac{p(p-1)}{2 \kappa_d^2} \int_0^2 g_{\mathbb{B}^d}(r) r^{p-2} \, dr \int_{S^{d-1}} \frac{\mathcal{H}^{d-1}(du)}{h_{\mathbb{E}(a)}(u)^{d+1-p}}.
\]

It remains to express the first integral in the last line by values of the \( \Gamma \)-function. Arguing from a purely geometric view point (including Cavalieri’s principle), see [6] (p. 326), we find that

\[
\int_0^2 g_{\mathbb{B}^d}(r) r^{p-2} \, dr = 2 \kappa_{d-1} \int_0^{r/2} \int_0 \left( \sqrt{1-x^2} \right)^{d-1} dx \, r^{p-2} \, dr
\]

\[
= \frac{2^{p-1} \kappa_{d-1}}{p-1} \int_0^1 r^{p/2-1} (1-r)^{(d+1)/2-1} \, dr = \frac{2^{p-1} \kappa_{d-1}}{p-1} \frac{B(p/2, (d+1)/2)}{2}.
\]
with Euler’s Beta-function $B(a, b)$ satisfying $B(a, b) = \Gamma(a) \Gamma(b)/\Gamma(a + b)$.

Summarizing the foregoing steps confirms (10) for $p > 1$. The case $p \in [0, 1]$ must be considered separately. From a formal point of view the term $p - 1$ occurring in the above formulas disappears by cancelling and in view of the relation $\Gamma(p/2) p/2 = \Gamma(p/2 + 1)$ the r.h.s. of (10) makes sense for $p = 0$ (and even for $p > -2$). To be rigorous we consider the r.h.s.’s of (1) and (10) as function of $p$ on the open interval $(0, 1 + \epsilon)$ for $\epsilon > 0$. It can be shown that both integrals are analytic functions on this interval and right-continuous in $p = 0$. Hence, both functions coincide and (10) holds for any $p \geq 0$. \hfill \Box

4 Alternative Calculation of CPI’s for Ellipses

Another way to calculate $\mathcal{I}_p(\mathbb{E}(a))$ consists in a direct evaluation of the double integral (1). For ellipses $\mathbb{E}(a, b) = \{(x, y) : x^2/a^2 + y^2/b^2 \leq 1\}$ with $a \geq b > 0$ we will sketch this. For this purpose let $u(\varphi) = (\cos \varphi, \sin \varphi)$ and assume w.l.o.g. that $0 \leq \varphi \leq \pi/2$. To facilitate the integration over the projection interval $\mathbb{E}(a, b)|u(\varphi)^\perp$ we rotate $u(\varphi)$ and $\mathbb{E}(a, b)$ anti-clockwise by the angle $\hat{\varphi} := \pi/2 - \varphi$. The rotation matrix $R(\varphi) = (r_{ij})_{i,j=1}^2$ with $r_{11} = -r_{22} = \sin \vartheta, r_{21} = -r_{12} = \cos \vartheta$ and the support functions $h(\varphi) = \sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi}$ and $h(\hat{\varphi}) = \sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}$ enables us to express the chord length in direction $u_0 = (0, 1)$ of the rotated ellipse

$R(\varphi)\mathbb{E}(a, b) = \{(x, y) : x^2 h^2(\varphi) + y^2 h^2(\hat{\varphi}) - \sin(2 \varphi) (a^2 - b^2) x y \leq a^2 b^2\}$

as follows:

$\mathcal{H}^1(R(\varphi)\mathbb{E}(a, b) \cap \ell(x, u_0)) = 2ab \sqrt{h^2(\hat{\varphi}) - x^2/h^2(\varphi)}$

for $x \in [-h(\hat{\varphi}), h(\hat{\varphi})]$.

For reasons of symmetry we can reduce the integral over $S^1$ on the r.h.s. of (1) to four integrals over the quadrant $[0, \pi/2]$ leading to

$$\mathcal{I}_p(\mathbb{E}(a, b)) = 2 \int_0^{\pi/2} \int_{-h(\hat{\varphi})}^{h(\varphi)} \left( \frac{2ab}{h(\hat{\varphi})} \left(1 - \frac{x^2}{h^2(\varphi)}\right)^{1/2}\right)^p \, dx \, d\varphi.$$ 

After a short calculation, where $h(\hat{\varphi})$ can be replaced by $h(\varphi)$, we arrive at

$$\mathcal{I}_p(\mathbb{E}(a, b)) = 2^{p+1} \sqrt{\pi} (a b)^p \frac{\Gamma\left(\frac{p+2}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)} \int_0^{\pi/2} \frac{d\varphi}{h(\varphi)^{p-1}} \text{ for any } p \geq 0. \quad (12)$$

It should be noted that (12) was given (without proof) in [14] for $p = 0, 1, \ldots$, see also [6] for $p = 2$, and the chord length distribution function $F_{\mu,\mathbb{E}(a, b)}$ has been derived in [10]. Equating (10) for $d = 2$ and (12) provides relations for complete elliptic integrals which are of interest in their own right.
Finally, we say a few words in an attempt to use (3) to calculate $\mathcal{I}_{k+1}(E(a))$. For doing this, we need formulas for the $k$-volume of the intersection $k$-ellipsoids $E(a)\cap L$ for $L \in \mathcal{A}(d,k)$. For $k = d - 1$ we succeeded in obtaining such a formula by exploiting the fact that the $(d-1)$-ellipsoids $E(p,u) := E(a) \cap (u^1 + pu)$ are homothetic for distinct $p \in [-h_{E(a)}(u), h_{E(a)}(u)]$. Finally, after some further rearrangements we arrive at

$$\mathcal{H}^{d-1}(E(p,u)) = \frac{\kappa_{d-1} V(E(a))}{\kappa_d h_{E(a)}(u)} \left( 1 - \frac{p^2}{h_{E(a)}(u)^2} \right)^{(d-1)/2}.$$  

This inserted in (3) for $k = d - 1$ shows that $\mathcal{I}_d(E(a))/\sqrt{V(E(a))}$ is equal to

$$\int_{S^{d-1}} \mathcal{H}^{d-1}(E(p,u))^2 \ dp \ \mathcal{H}^{d-1}(du) = \frac{2^{d-1} \kappa_2 d-1}{\kappa_d^3} \int_{S^{d-1}} \mathcal{H}^{d-1}(du) \ / h_{E(a)}(u).$$

5 Some Sharp Estimates for CPI’s of Ellipsoids

Hölder’s inequality applied to the moments $EL_{\mu,L}^k$ in Remark 1 implies that

$$m_{k+1}(K) \geq m_i(K) m_j(K) \geq m_1(K)^{k+1} = (d \kappa_d V(K)/\kappa_{d-1} S(K))^{k+1}$$

for $i + j = k + 1$ and $k = 0, 1, \ldots$. Putting $i = 1, j = k$ gives the inequality

$$\frac{\mathcal{I}_{k+1}(K)}{\mathcal{I}_k(K)} \geq \frac{\mathcal{I}_1(K)}{\mathcal{I}_0(K)} = \frac{d \kappa_d V(K)}{\kappa_{d-1} S(K)} \text{ for } k = 0, 1, \ldots , \text{ see } [11] \ (p. 48), \quad (13)$$

where “$=$” on the l.h.s. is impossible for $k \geq 1$ since $\mathcal{P}(L_{\mu,K} = \text{const}) < 1$.

P.J. Davy [3] posed the following considerable improvement of (13) as unsolved question. Just for $k \in \{d,d+1\}$ she gave a positive answer in [3].

Conjecture 1. For any convex body $K \subset \mathbb{R}^d$ with inner points and $k \geq 0$,

$$\frac{\mathcal{I}_{k+1}(K)}{\mathcal{I}_k(K)} \geq \frac{2 \kappa_k \kappa_{d-1} \kappa_{d+k}}{\kappa_k+1 \kappa_d \kappa_{d+k-1}} \frac{\mathcal{I}_1(K)}{\mathcal{I}_0(K)} = \frac{2 d \kappa_k \kappa_{d+k}}{\kappa_k+1 \kappa_d \kappa_{d+k-1}} \frac{V(K)}{S(K)} \quad (14)$$

with “$=$” being attained only for balls, where $\frac{2 \kappa_k \kappa_{d-1} \kappa_{d+k}}{\kappa_k+1 \kappa_d \kappa_{d+k-1}} > 1$ for $k \geq 1$.

To get an explicit lower bound of $\mathcal{I}_{k+1}(K)$ (as conjecture) we multiply on both sides of (14) over $k = 1, \ldots , n$ (after that we set $n = k$) and replace $\mathcal{I}_1(K)$ by $d \kappa_d V(K)/2$. In this way together with (7) we get the inclusion

$$2^k d^{k+1} \frac{V(K)^{k-1}}{S(K)^k} \leq \mathcal{I}_{k+1}(K) \leq \frac{2^k d \kappa_{d+k}}{\kappa_k+1} \left( \frac{V(K)}{\kappa_d} \right)^{k/d} \quad (15)$$
for $k = 1, \ldots, d$, where for $k = 0$ both sides of (15) are trivially true; the lower bound is proved for $k = d, d+1$, but still open for $k = 1, \ldots, d-1$.

It should be noticed the fact that the inequality induced by the lower and upper bound in (15) coincides with the well-known isoperimetric inequality

$$d \frac{V(K)}{S(K)} \leq \left( \frac{V(K)}{\kappa_d} \right)^{1/d}, \quad \text{where} \quad \text{“} = \text{” holds iff } K \text{ is a ball}. \quad (16)$$

This means that the validity of (15) would present an (almost) optimal estimate of $I_k(K)$ and slightly strengthen the isoperimetric inequality.

In what follows we prove the inclusion (15) for $K = \mathbb{E}(a)$ and derive in this case at least for $k = d-1$ slightly larger lower bound in terms of $b(\mathbb{E}(a))$.

**Theorem 5.1** For the ellipsoid $K = \mathbb{E}(a)$ defined in (8) the Conjecture 1 and therefore the inclusion (15) are true. Furthermore, the inclusion

$$C_{d,p} \left( \frac{2}{b(\mathbb{E}(a))} \right)^{d+1-p} \leq I_p(\mathbb{E}(a))/V^2(\mathbb{E}(a)) \leq C_{d,p} \left( \frac{\kappa_d}{V(\mathbb{E}(a))} \right)^{(d+1-p)/d} \quad (17)$$

holds with $C_{d,p} = 2^{p-1} \pi^{-1/2} d \Gamma \left( \frac{d}{2} + 1 \right) \Gamma \left( \frac{p}{2} + 1 \right) / \Gamma \left( \frac{d+p+1}{2} \right)$ for any $p \in [1, d]$. For $p = k+1$ we may write $C_{d,k+1} = 2^k d \kappa_{d+k}/\kappa_{k+1} \kappa_d$ for $k = 0, 1, \ldots, d-1$.

**Proof.** First we express the ratio $I_{k+1}(\mathbb{E}(a))/I_k(\mathbb{E}(a))$ by means of (10) in terms of the function $Y(u) = 1/h_{\mathbb{E}(a)}(u)$ for $u \in S^{d-1}$. Setting $p = k+1$ resp. $p = k$ in (10) and replacing $\Gamma(n/2+1)$ by $\pi^{n/2}/\kappa_n$ for $n \in \{k, k+1, d-k, d-k+1\}$ yield the identity

$$\frac{I_{k+1}(\mathbb{E}(a))}{I_k(\mathbb{E}(a))} = \frac{2 \kappa_k \kappa_{d+k}}{\kappa_{k+1} \kappa_{d+k+1}} \int_{S^{d-1}} Y(u)^{d-k} \mathcal{H}^{d-1}(du) \int_{S^{d-1}} Y(u)^{d-k+1} \mathcal{H}^{d-1}(du). \quad (18)$$

The r.h.s. of (18) remains unchanged if $\mathcal{H}^{d-1}(\cdot)$ is replaced by the uniform distribution $U(\cdot) := \mathcal{H}^{d-1}(\cdot)/d \kappa_d$ on $S^{d-1}$. In this way we may write the ratio of the both integrals over $S^{d-1}$ in (18) as ratio $E_U Y^{d-k} / E_U Y^{d-k+1}$ of moments of the random variable $Y(\cdot)$ w.r.t. the uniform distribution $U$ on $S^{d-1}$. Putting $k = 0$ in (18) yields $I_1(\mathbb{E}(a))/I_0(\mathbb{E}(a)) = \kappa_d E_U Y^{d} / \kappa_{d-1} E_U Y^{d+1}$. Now, we insert this identity in the desired inequality (14) for $K = \mathbb{E}(a)$. Comparing the resulting relation (14) with (18) we deduce that (14) is equivalent with the moment inequality

$$\frac{E_U Y^d}{E_U Y^{d+1}} \leq \frac{E_U Y^{d-k}}{E_U Y^{d-k+1}} \quad \text{for any} \quad k = 1, 2, \ldots. \quad (19)$$
Since $Y$ is strictly positive and bounded, it easily seen by the Cauchy-Schwarz inequality that
\[
E_U Y^{d-j} \leq \sqrt{E_U Y^{d-j-1}} \sqrt{E_U Y^{d-j+1}} \quad \text{i.e.} \quad \frac{E_U Y^{d-j}}{E_U Y^{d-j+1}} \leq \frac{E_U Y^{d-j-1}}{E_U Y^{d-j}}
\]
inequality (10) for $j = 0, 1, \ldots, k - 1$ and $k = 1, 2, \ldots$, implying immediately (19). Note that \( \text{"=} \) holds iff $P(Y = h_{E(a)}^{-1} = \text{const}) = 1$, i.e., $a_1 = \cdots = a_d = \text{const}$.
Thus, the first part of Theorem 5.1 is proved. To prove the second part we start with $1 = (h_{E(a)}(u) Y(u))^{1/2}$ for $u \in S^{d-1}$ and apply the Cauchy-Schwarz and twice the Hölder inequality leading to
\[
1 \leq E_U h_{E(a)} E_U Y \leq E_U h_{E(a)} (E_U Y^{d+1-p})^{\frac{1}{p+1-p}} \leq E_U h_{E(a)} (E_U Y^d)^{\frac{1}{2}}
\]
for any real $p \in [1, d]$. Next we write $E_U$ as integral over $S^{d-1}$ w.r.t. uniform distribution $U$ and make use of the mean breadth $b(E(a)) = 2E_U h_{E(a)}$ (see in Sect. 3) with support function (9). This amounts to the inclusion
\[
\left( \frac{2}{b(E(a))} \right)^{d+1-p} \leq \int_{S^{d-1}} \frac{U(du)}{h_{E(a)}(u)^{d+1-p}} \leq \left( \int_{S^{d-1}} \frac{U(du)}{h_{E(a)}(u)^d} \right)^{(d+1-p)/d}. \tag{20}
\]
Combining (10) for $p = 1$ and $\mathcal{I}_p(K) = d \kappa_d V(K)/2$ reveals the remarkable relation
\[
\int_{S^{d-1}} \frac{U(du)}{h_{E(a)}(u)^d} = \frac{\kappa_d}{V(E(a))} = \frac{1}{a_1 \cdots a_d}. \tag{21}
\]
We mention that (21) can also be verified directly by using the spherical coordinates $u_1 = \cos \vartheta_1 \sin \vartheta_2 \cdots \sin \vartheta_{d-1}$, $u_2 = \sin \vartheta_1 \sin \vartheta_2 \cdots \sin \vartheta_{d-1}$ and $u_i = \cos \vartheta_{i-1} \prod_{j=1}^{i-1} \sin \vartheta_j$ for $i = 3, \ldots, d$ with infinitesimal surface element $\mathcal{H}^{d-1}(du) = \prod_{i=2}^{d-1} (\sin \vartheta_i)^{i-2} d(\vartheta_1, \vartheta_2, \ldots, \vartheta_{d-1})$, where $\vartheta_1 \in [0, 2\pi]$ and $\vartheta_i \in [0, \pi]$ for $i = 2, \ldots, d - 1$.

Finally, we multiply the inclusion (20) by the constant $C_{d,p}$ which is chosen in view of (10) such that the middle term multiplied by $C_{d,p}$ just equals $\mathcal{I}_p(E(a))/V(E(a))^2$. This completes the proof of Theorem 5.1.

**Remark 2.** From (17) we get Urysohn's inequality, see [1], for $E(a)$, namely,
\[
2 \left( \frac{V(E(a))}{\kappa_d} \right)^{1/d} \leq b(E(a)) = \frac{2}{d \kappa_d} \int_{S^{d-1}} h_{E(a)}(u) \mathcal{H}^{d-1}(du)
\]
with \( \text{"=} \) iff $a_1 = \cdots = a_d$. By comparison of the inclusions (15) for $K = E(a)$ and (17) for $p = k + 1$ we see that the upper bounds coincide but the lower bound in (17) for $k = d - 1$ (resp. for $k = 1$) is $\geq$ (resp. $\leq$) the lower bound in (15) due to the Aleksandrov-Fenchel-type inequality $2 V(K) (S(K)/d V(K))^{d-1} \geq \kappa_d b(K)$ (resp. the isoperimetric-type inequality $b(K) \geq 2 (S(K)/d \kappa_d)^{1/(d-1)}$).
This gives rise to formulate the following alternative to (15):

**Conjecture 2.** For any convex body \( K \subset \mathbb{R}^d \) with inner points and \( k \geq 0 \),

\[
\frac{2^k d^k \kappa_{d+k}}{\kappa_d} \left( \frac{2}{b(K)} \right)^{d-k} \leq \frac{I_{k+1}(K)}{V(K)^2} \leq \frac{2^k d^k \kappa_{d+k}}{\kappa_d} \left( \frac{\kappa_d}{V(K)} \right)^{(d-k)/d} \tag{22}
\]

with “=” being attained only for balls, see [6] for the special case \( k = d - 1 \).

## 6 CPI’s of Superellipsoids and Simplices

We conclude this paper with a few remarks on CPI’s of two important classes of convex bodies in \( \mathbb{R}^d \): superellipsoids \( \mathbb{E}_\alpha(a) \) and simplices \( S(P) \). To be precise, we start with a definition of these bodies. In generalizing (8) the convex body \( \mathbb{E}_\alpha(a) := \{ x \in \mathbb{R}^d : (|x_1|/a_1)^\alpha + \cdots + (|x_d|/a_d)^\alpha \leq 1 \} \) is called a superellipsoid \( \mathbb{E}_\alpha(a) \) of degree \( \alpha \geq 1 \) with semi-axes \( a = (a_1, \ldots, a_d) \) and \( S(P) \) is defined to be the convex hull of the \( d + 1 \) points \( P = \{ o, p_1, \ldots, p_d \} \), where the vectors \( p_1, \ldots, p_d \) are linearly independent.

Special cases are the cross-polytope \( \mathbb{E}_1(a) \), the ellipsoid \( \mathbb{E}(a) \) for \( \alpha = 2 \) and the hyper-rectangle \( \mathbb{E}_\infty(a) = \times_{i=1}^d [-a_i, a_i] \) all of them are \( o \)-symmetric with support function \( h_{\mathbb{E}_\alpha}(u) = (|u_1/a_1|^\alpha + \cdots + |u_d/a_d|^\alpha)^{1/\beta} \) with \( \beta = \alpha/(\alpha - 1) \) for \( \alpha > 1 \) and \( h_{\mathbb{E}_1}(u) = \max \{|u_1|, \ldots, |u_d| a_d\} \).

By introducing the \( \ell_\alpha \)-norm \( ||x||_\alpha = (|x_1|^\alpha + \cdots + |x_d|^\alpha)^{1/\alpha} \) for \( 1 \leq \alpha < \infty \) and \( ||x||_\infty := \max \{|x_1|, \ldots, |x_d|\} \) we define the unit ball \( \mathbb{B}_\alpha^d := \{ x \in \mathbb{R}^d : ||x||_\alpha \leq 1 \} \) and its volume \( \kappa_{d,\alpha} := V(\mathbb{B}_\alpha^d) \) w.r.t. the \( \ell_\alpha \)-norm. Since \( \mathbb{E}_\alpha(a) = \mathbb{A} \mathbb{B}_\alpha^d \) with diagonal matrix \( A = \text{diag}[a_1, \ldots, a_d] \) it follows that \( V(\mathbb{E}_\alpha(a)) = \kappa_{d,\alpha} \prod_{i=1}^d a_i \).

By repeating the proof of Theorem 1 (which is left to the reader) we can generalize (10) as follows:

**Theorem 6.1** It holds that \( I_1(\mathbb{E}_\alpha(a)) = \frac{d}{2} \kappa_{d,\alpha} \prod_{i=1}^d a_i \) and, for \( p > 1 \),

\[
I_p(\mathbb{E}_\alpha(a)) = \frac{p(p-1)}{2 \kappa_{d,\alpha}^2} \int_{\partial \mathbb{B}_\alpha^d} \int_0^{2/\|u\|_\alpha} \frac{V(\mathbb{E}_\alpha^d \cap (\mathbb{E}_\alpha^d + r u))}{h_{\mathbb{E}_\alpha}(u)^{d+1-p}} r^{p-2} \, dr \, \mathcal{H}^{d-1}(du)
\]

for arbitrary \( \gamma \in [1, \infty] \), where \( 2/\|u\|_\alpha = \inf \{ r > 0 : \mathbb{B}_\alpha^d \cap (\mathbb{B}_\alpha^d + r u) = \emptyset \} \).

To conclude with, we quote a result on CPI’s of \( S(P) \) proved in [15] (p.593)

\[
\frac{I_p(S(P))}{V(S(P))} = \frac{d!}{2 \Gamma(p+1)} \int_{\mathcal{H}^{d-1}} \left( \frac{d V(S(P))}{\mathcal{H}^{d-1}(S(P)|u^*)} \right)^{p-1} \, du \tag{23}
\]
for any \( p \geq 1 \) with the well-known formula \( V(S(P)) = |\det(p_1, \ldots, p_d)|/d! \).

An astonishing relation proved in [8] says that
\[
\mathcal{H}^{d-1}(S(P)|u^\perp) \rho_{DS}(P)(u) = dV(S(P)) \forall u \in S^{d-1},
\]
where \( \rho_{DK}(u) := \max\{\lambda > 0 : \lambda u \in K \oplus (-K)\} \)
denotes the radial function of the difference body \( DK := K \oplus (-K) \) of \( K \). It is not difficult to show that \( \rho_{DK}(u) = \max\{\mathcal{H}^1(K \cap \ell(x, u)) : x \in u^\perp\} \) = length of the longest chord of \( K \) in direction \( u \). In view of this geometric relation the integrand on the r.h.s. of (23) can be replaced by \( \rho_{DS}(P)(u)p^{-1} \), see also [2] for a different approach.

**References**


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