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Generalized Fibonacci and Lucas Sums from Residue Classes of the Number 3

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Dedicated to Professor Marius Stoka on the occasion of his 80^{th} *birthday*

Abstract

In this paper we consider some sums of generalized Fibonacci and Lucas numbers U_n and V_n when their indices n belong to one of the three residue classes of the number 3. The explicit formulas for these sums that we discover are products or very close to products.

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1. Introduction

In this paper we consider some types of finite sums of generalized Fibonacci and Lucas numbers U_n and V_n introduced by Lucas in [5]. The main feature of these sums is that the indices of summands are from the same residue class of the number 3 (i. e., we take every third term), that the sums are quite simple products or very close to such products and that often the form of the summation formula depends on the residue class of the number 4 to which the numbers of summands are from. These formulas could all be proved by mathematical induction. We leave these proofs to the reader as a challenge. Another method of proof uses Binet forms of numbers U_n and V_n and the formulas for sums of geometric series. With this approach we shall prove three identities. The others are also left to the reader.

There are many nice summation formulas for (generalized) Fibonacci and Lucas numbers in the literature (see, for example, [1], [2], [3], [4], [6], [7], [8], [9], [10] and [11]). We hope that the readers will find the ones that follow also interesting.

For any natural number k, let $p_k = p - k$, $q_k = p + k$, $S_k = p^2 - k$ and $T_k = p^2 + k$. Let $\Delta = T_4$.

2. Some sums that are products

Our first theorem considers sums of generalized Fibonacci and Lucas numbers with indices that are multiples of the number three. The form of these summation formulas depend on how many consecutive terms we take. More precisely, it depends on the residue class modulo 4 of the number of terms in the sum.

Theorem 1. *Let $A = p\,T_3$. For all integers $i \geq 0$, we have*

$$(1.1)\quad A \sum_{j=0}^{4i} U_{3j} = U_{6i}\left(V_{6i+3} + V_{6i}\right),$$

$$(1.2)\quad A \sum_{j=0}^{4i} (-1)^j\, U_{3j} = U_{6i}\left(V_{6i+3} - V_{6i}\right),$$

$$(1.3)\quad A \sum_{j=0}^{4i+1} U_{3j} = V_{6i+3}\left(q_1\, U_{6i+2} - p_1\, U_{6i+1}\right),$$

$$(1.4)\quad A \sum_{j=0}^{4i+1} (-1)^j\, U_{3j} = -V_{6i+3}\left(p_1\, U_{6i+2} + q_1\, U_{6i+1}\right),$$

$$(1.5)\quad A \sum_{j=0}^{4i+2} U_{3j} = V_{6i+3}\left(q_1\, U_{6i+5} - p_1\, U_{6i+4}\right),$$

$$(1.6)\quad A \sum_{j=0}^{4i+2} (-1)^j\, U_{3j} = V_{6i+3}\left(p_1\, U_{6i+5} + q_1\, U_{6i+4}\right),$$

$$(1.7)\quad A \sum_{j=0}^{4i+3} U_{3j} = U_{6i+6}\left(V_{6i+3} + V_{6i+6}\right),$$

$$(1.8)\quad A \sum_{j=0}^{4i+3} (-1)^j\, U_{3j} = U_{6i+6}\left(V_{6i+3} - V_{6i+6}\right),$$

$$(1.9)\quad A \sum_{j=1}^{4i} V_{3j} = \Delta\, U_{6i}\left(q_1\, U_{6i+2} - p_1\, U_{6i+1}\right),$$

$$(1.10) \quad A\sum_{j=1}^{4i}(-1)^j\,V_{3j} = \Delta\,U_{6i}\,(p_1\,U_{6i+2}+q_1\,U_{6i+1})\,,$$

$$(1.11) \quad A\sum_{j=0}^{4i+1}V_{3j} = V_{6i+3}\,(V_{6i}+V_{6i+3})\,,$$

$$(1.12) \quad A\sum_{j=0}^{4i+1}(-1)^j\,V_{3j} = V_{6i+3}\,(V_{6i}-V_{6i+3})\,,$$

$$(1.13) \quad A\sum_{j=1}^{4i+2}V_{3j} = V_{6i+3}\,(q_1\,V_{6i+5}-p_1\,V_{6i+4})\,,$$

$$(1.14) \quad A\sum_{j=1}^{4i+2}(-1)^j\,V_{3j} = V_{6i+3}\,(p_1\,V_{6i+5}+q_1\,V_{6i+4})\,,$$

$$(1.15) \quad A\sum_{j=0}^{4i+3}V_{3j} = \Delta\,U_{6i+6}\,(U_{6i+3}+U_{6i+6})\,,$$

$$(1.16) \quad A\sum_{j=0}^{4i+3}(-1)^j\,V_{3j} = \Delta\,U_{6i+6}\,(U_{6i+3}-U_{6i+6})\,.$$

Proof of (1.1). Let p be an arbitrary complex number such that $p \neq 0$ and $p^2 \neq -2, -4$. The roots $\alpha = \frac{p+\sqrt{\Delta}}{2}$ and $\beta = \frac{p-\sqrt{\Delta}}{2}$ of the equation

$$z^2 - p\,z - 1 = 0$$

are distinct , where $\Delta = p^2 + 4$. The sequences U_n and V_n are given by their Binet forms as $U_n = \frac{\alpha^n - \beta^n}{\alpha-\beta}$, $V_n = \alpha^n + \beta^n$ for any integer n.

Notice that $A = p(p^2+3) = (\alpha+\beta)[(\alpha+\beta)^2+3]$ is equal to

$$\beta^3 + 3\,\beta^2\alpha + 3\,\beta\,\alpha^2 + 3\,\beta + \alpha^3 + 3\,\alpha = \beta^3 + \alpha^3,$$

where we used the identity $\alpha\beta = -1$ in the terms containing both α and β.

The left hand side LHS of (1.1) is equal

$$\frac{(\alpha^3+\beta^3)(\beta^{12\,i} - \alpha^{12\,i} - \alpha^{12\,i+3} + \beta^{12\,i+3} - \beta^3 + \alpha^3)}{B},$$

where $B = (\alpha-\beta)\,(\alpha^3-1)\,(\beta^3-1)$.

The right hand side RHS of (1.1) is equal

$$\frac{(\alpha^{6\,i} - \beta^{6\,i})\,(\alpha^{6\,i+3} + \beta^{6\,i+3} + \alpha^{6\,i} + \beta^{6\,i})}{\alpha-\beta}.$$

The difference $LHS - RHS$ is $\frac{C}{B}$, where C is

$$\beta^{6\,i}\alpha^{6\,i+3} - \alpha^{6\,i}\beta^{6\,i+3} + \alpha^{12\,i}\beta^{3} - \alpha^{12\,i+3} + \beta^{12\,i+3} + \alpha^{12\,i+6} - \beta^{12\,i+6} + \beta^{12\,i+6}\alpha^{3} - \alpha^{12\,i+6}\beta^{3} + \alpha^{6\,i}\beta^{6\,i+6} - \beta^{6\,i}\alpha^{6\,i+6} + \alpha^{3} - \beta^{3} - 2\,\alpha^{12\,i} - \beta^{12\,i}\alpha^{3} + 2\,\beta^{12\,i} + \beta^{6\,i+3}\alpha^{6\,i+6} - \alpha^{6\,i+3}\beta^{6\,i+6}$$

When we make the reduction of terms containing both α and β using $\alpha\beta = -1$ an easy conclusion is that $C = 0$ so that $LHS = RHS$. □

Analogous results hold for the sums of generalized Fibonacci and Lucas numbers whose indices are numbers having the reminder one when divided by the number three.

Theorem 2. *Let $A = p\,T_3$. For all integers $i \geq 0$, we have*

$$A\sum_{j=0}^{4i} U_{3j+1} = V_{6i+1}\left(U_{6i+3} + U_{6i}\right) + 2\,p, \tag{2.1}$$

$$A\sum_{j=0}^{4i} (-1)^j\, U_{3j+1} = V_{6i+1}\left(U_{6i+3} - U_{6i}\right) + 2\,p, \tag{2.2}$$

$$A\sum_{j=0}^{4i+1} U_{3j+1} = V_{6i+3}\left(U_{6i+1} + U_{6i+4}\right), \tag{2.3}$$

$$A\sum_{j=0}^{4i+1} (-1)^j\, U_{3j+1} = V_{6i+3}\left(U_{6i+1} - U_{6i+4}\right), \tag{2.4}$$

$$A\sum_{j=0}^{4i+2} U_{3j+1} = U_{6i+4}\left(V_{6i+6} + V_{6i+3}\right) + 2\,p, \tag{2.5}$$

$$A\sum_{j=0}^{4i+2} (-1)^j\, U_{3j+1} = U_{6i+4}\left(V_{6i+6} - V_{6i+3}\right) + 2\,p, \tag{2.6}$$

$$A\sum_{j=0}^{4i+3} U_{3j+1} = U_{6i+6}\left(V_{6i+4} + V_{6i+7}\right), \tag{2.7}$$

$$A\sum_{j=0}^{4i+3} (-1)^j\, U_{3j+1} = U_{6i+6}\left(V_{6i+4} - V_{6i+7}\right), \tag{2.8}$$

$$A\sum_{j=0}^{4i} V_{3j+1} = V_{6i+1}\left(V_{6i+3} + V_{6i}\right) - 2\,p, \tag{2.9}$$

$$A\sum_{j=0}^{4i} (-1)^j\, V_{3j+1} = V_{6i+1}\left(V_{6i+3} - V_{6i}\right) + 2\,p, \tag{2.10}$$

$$(2.11)\quad A\sum_{j=0}^{4i+1} V_{3j+1} = V_{6i+3}\left(V_{6i+1}+V_{6i+4}\right),$$

$$(2.12)\quad A\sum_{j=0}^{4i+1} (-1)^j\, V_{3j+1} = V_{6i+3}\left(V_{6i+1}-V_{6i+4}\right),$$

$$(2.13)\quad A\sum_{j=0}^{4i+2} V_{3j+1} = V_{6i+4}\left(V_{6i+6}+V_{6i+3}\right)-2\,T_2,$$

$$(2.14)\quad A\sum_{j=0}^{4i+2} (-1)^j\, V_{3j+1} = V_{6i+4}\left(V_{6i+6}-V_{6i+3}\right)-2\,T_2,$$

$$(2.15)\quad A\sum_{j=0}^{4i+3} V_{3j+1} = V_{6i+6}\left(V_{6i+4}+V_{6i+7}\right)-2\left(T_2+p\right),$$

$$(2.16)\quad A\sum_{j=0}^{4i+3} (-1)^j\, V_{3j+1} = V_{6i+6}\left(V_{6i+4}-V_{6i+7}\right)-2\left(T_2-p\right).$$

Proof of (2.3). The left hand side LHS of (2.3) is equal

$$\frac{(\alpha^3+\beta^3)(\beta^{12\,i+4}-\alpha^{12\,i+4}-\alpha^{12\,i+7}+\beta^{12\,i+7}+\beta^2-\alpha^2+\alpha-\beta)}{B}.$$

The right hand side RHS of (2.3) is equal

$$\frac{(\alpha^{6\,i+3}+\beta^{6\,i+3})\,(\alpha^{6\,i+1}-\beta^{6\,i+1}+\alpha^{6\,i+4}-\beta^{6\,i+4})}{\alpha-\beta}.$$

The difference $LHS-RHS$ is $\frac{D}{B}$, where D is

$$\alpha^{12\,i+7}-\beta^{12\,i+7}-\beta^{6\,i+3}\alpha^{6\,i+1}+\alpha^{12\,i+10}-\beta^{12\,i+10}+\beta- \\ \alpha-\alpha^{6\,i+3}\beta^{6\,i+7}+\alpha^2-\beta^2+\alpha^{6\,i+3}\beta^{6\,i+1}-\beta^{12\,i+4}\alpha^3+\alpha^{12\,i+4}\beta^3- \\ \beta^{6\,i+6}\alpha^{6\,i+7}-\alpha^{6\,i+6}\beta^{6\,i+1}+\alpha^{6\,i+6}\beta^{6\,i+7}+\alpha^3\beta^{12\,i+10}-\alpha^{12\,i+10}\beta^3+ \\ \beta^{6\,i+6}\alpha^{6\,i+1}+\beta^{6\,i+3}\alpha^{6\,i+7}.$$

Again, the reduction of terms containing both α and β using $\alpha\beta=-1$ gives the conclusion that $D=0$ so that $LHS=RHS$. □

We have similar formulas for the (alternating) sums of generalized Fibonacci and Lucas numbers with indices that are numbers having the reminder two when divided by the number three.

Theorem 3. *Let $A=p\,T_3$. For all integers $i\geq 0$, we have*

$$(3.1)\quad A\sum_{j=0}^{4i} U_{3j+2} = V_{6i+1}\left(U_{6i+4}+U_{6i+1}\right)+p_1\,p,$$

$$A\sum_{j=0}^{4i}(-1)^j\,U_{3j+2}=V_{6i+1}\left(U_{6i+4}-U_{6i+1}\right)+q_1\,p, \tag{3.2}$$

$$A\sum_{j=0}^{4i+1}U_{3j+2}=V_{6i+3}\left(U_{6i+2}+U_{6i+5}\right), \tag{3.3}$$

$$A\sum_{j=0}^{4i+1}(-1)^j\,U_{3j+2}=V_{6i+3}\left(U_{6i+2}-U_{6i+5}\right), \tag{3.4}$$

$$A\sum_{j=0}^{4i+2}U_{3j+2}=U_{6i+4}\left(V_{6i+7}+V_{6i+4}\right)+p_1\,p, \tag{3.5}$$

$$A\sum_{j=0}^{4i+2}(-1)^j\,U_{3j+2}=U_{6i+4}\left(V_{6i+7}-V_{6i+4}\right)+q_1\,p, \tag{3.6}$$

$$A\sum_{j=0}^{4i+3}U_{3j+2}=U_{6i+6}\left(V_{6i+5}+V_{6i+8}\right), \tag{3.7}$$

$$A\sum_{j=0}^{4i+3}(-1)^j\,U_{3j+2}=U_{6i+6}\left(V_{6i+5}-V_{6i+8}\right), \tag{3.8}$$

$$A\sum_{j=0}^{4i}V_{3j+2}=V_{6i+1}\left(V_{6i+4}+V_{6i+1}\right)+p\left(\Delta-p\right), \tag{3.9}$$

$$A\sum_{j=0}^{4i}(-1)^j\,V_{3j+2}=V_{6i+1}\left(V_{6i+4}-V_{6i+1}\right)+p\left(\Delta+p\right), \tag{3.10}$$

$$A\sum_{j=0}^{4i+1}V_{3j+2}=V_{6i+3}\left(V_{6i+2}+V_{6i+5}\right), \tag{3.11}$$

$$A\sum_{j=0}^{4i+1}(-1)^j\,V_{3j+2}=V_{6i+3}\left(V_{6i+2}-V_{6i+5}\right), \tag{3.12}$$

$$A\sum_{j=0}^{4i+2}V_{3j+2}=V_{6i+4}\left(V_{6i+7}+V_{6i+4}\right)-p\,T_2-\Delta, \tag{3.13}$$

$$A\sum_{j=0}^{4i+2}(-1)^j\,V_{3j+2}=V_{6i+4}\left(V_{6i+7}-V_{6i+4}\right)-p\,T_2+\Delta, \tag{3.14}$$

$$A\sum_{j=0}^{4i+3}V_{3j+2}=V_{6i+6}\left(V_{6i+5}+V_{6i+8}\right)-2\left(T_2-p\right), \tag{3.15}$$

$$(3.16) \quad A \sum_{j=0}^{4i+3} (-1)^j \, V_{3j+2} = V_{6i+6} \, (V_{6i+5} - V_{6i+8}) + 2 \, (T_2 + p) \, .$$

Proof of (3.8). The left hand side LHS of (3.8) is equal

$$\frac{(\alpha^3 + \beta^3)(\alpha^{12\,i+14} - \beta^{12\,i+14} - \alpha^{12\,i+11} + \beta^{12\,i+11} + \alpha + \beta^2 - \beta - \alpha^2)}{B'},$$

where $B' = (\alpha - \beta)\,(\alpha^3 + 1)\,(\beta^3 + 1)$.

The right hand side RHS of (3.8) is equal

$$\frac{(\alpha^{6\,i+6} - \beta^{6\,i+6})\,(\alpha^{6\,i+5} + \beta^{6\,i+5} - \alpha^{6\,i+8} - \beta^{6\,i+8})}{\alpha - \beta}.$$

The difference $LHS - RHS$ is $\frac{E}{B'}$, where E is

$$\begin{aligned}&\alpha^{6\,i+6}\beta^{6\,i+11} - \beta + \alpha - \beta^{6\,i+6}\alpha^{6\,i+11} + \beta^{6\,i+6}\alpha^{6\,i+5} - \alpha^{6\,i+6}\beta^{6\,i+5} - \\ &\quad \alpha^2 + \beta^2 - \beta^{6\,i+9}\alpha^{6\,i+11} + \beta^{12\,i+11}\alpha^3 - \alpha^{12\,i+11}\beta^3 - \alpha^3\beta^{12\,i+17} - \\ &\quad\quad 2\,\alpha^{12\,i+11} + 2\,\beta^{12\,i+11} + \alpha^{12\,i+17}\beta^3 - \beta^{12\,i+17} + \alpha^{12\,i+17} + \\ &\quad\quad\quad \alpha^{12\,i+14} - \beta^{12\,i+14} + \beta^{6\,i+9}\alpha^{6\,i+5} + \alpha^{6\,i+9}\beta^{6\,i+11} - \alpha^{6\,i+9}\beta^{6\,i+5}.\end{aligned}$$

Once again the reduction of terms containing both α and β using $\alpha\beta = -1$ gives the conclusion that $E = 0$ so that $LHS = RHS$. □

3. Sums of products

The sums of consecutive products $x_u \, y_v$ of generalized Fibonacci and Lucas numbers with indices u and v from the residue classes 3_0, 3_1 and 3_2 of the number three are either themselves products or are rather close to them. The first result covers the case when $u \in 3_0$ and $v \in 3_1$.

Theorem 4. *For all integers $i \geq 0$, we have*

$$(4.1) \quad A \sum_{j=0}^{4i} U_{3j} \, U_{3j+1} = U_{12i} \, U_{12i+4},$$

$$(4.2) \quad A \sum_{j=0}^{4i} U_{3j} \, V_{3j+1} = A \sum_{j=0}^{4i} V_{3j} \, U_{3j+1} = V_{12i} \, U_{12i+4} - 2 \, p \, T_2,$$

$$(4.3) \quad A \sum_{j=0}^{4i+1} U_{3j} \, U_{3j+1} = U_{12i+4} \, U_{12i+6},$$

$$(4.4) \quad A \sum_{j=0}^{4i+1} U_{3j} \, V_{3j+1} = A \sum_{j=0}^{4i+1} V_{3j} \, U_{3j+1} = V_{12i+4} \, U_{12i+6},$$

$$(4.5) \quad A \sum_{j=0}^{4i+2} U_{3j} \, U_{3j+1} = U_{12i+6} \, U_{12i+10},$$

$$(4.6)\quad A\sum_{j=1}^{4i+2} U_{3j}\,V_{3j+1} = A\sum_{j=1}^{4i+2} V_{3j}\,U_{3j+1} = V_{12i+10}\,U_{12i+6},$$

$$(4.7)\quad A\sum_{j=0}^{4i+3} U_{3j}\,U_{3j+1} = U_{12i+10}\,U_{12i+12},$$

$$(4.8)\quad A\sum_{j=0}^{4i+3} U_{3j}\,V_{3j+1} = A\sum_{j=0}^{4i+3} V_{3j}\,U_{3j+1} = V_{12i+10}\,U_{12i+12}.$$

The second result covers the case when $u \in 3_1$ and $v \in 3_2$.

Theorem 5. *For all integers $i \geq 0$, we have*

$$(5.1)\quad A\sum_{j=1}^{4i} U_{3j+1}\,U_{3j+2} = U_{12i}\,U_{12i+6},$$

$$(5.2)\quad A\sum_{j=1}^{4i} U_{3j+1}\,V_{3j+2} = A\sum_{j=1}^{4i} V_{3j+1}\,U_{3j+2} = U_{12i}\,V_{12i+6},$$

$$(5.3)\quad A\sum_{j=0}^{4i+1} U_{3j+1}\,U_{3j+2} = U_{12i+6}^2,$$

$$(5.4)\quad A\sum_{j=0}^{4i+1} U_{3j+1}\,V_{3j+2} = A\sum_{j=0}^{4i+1} V_{3j+1}\,U_{3j+2} = U_{24i+12},$$

$$(5.5)\quad A\sum_{j=1}^{4i+2} U_{3j+1}\,U_{3j+2} = U_{12i+6}\,U_{12i+12},$$

$$(5.6)\quad A\sum_{j=1}^{4i+2} U_{3j+1}\,V_{3j+2} = A\sum_{j=1}^{4i+2} V_{3j+1}\,U_{3j+2} = U_{12i+6}\,V_{12i+12},$$

$$(5.7)\quad A\sum_{j=0}^{4i+3} U_{3j+1}\,U_{3j+2} = U_{12i+12}^2,$$

$$(5.8)\quad A\sum_{j=0}^{4i+3} U_{3j+1}\,V_{3j+2} = A\sum_{j=0}^{4i+3} V_{3j+1}\,U_{3j+2} = U_{24i+24}.$$

Finally, the third result considers the case when $u \in 3_0$ and $v \in 3_2$.

Theorem 6. *For all integers $i \geq 0$, we have*

$$(6.1)\quad A\sum_{j=1}^{4i} U_{3j}\,U_{3j+2} = U_{12i}\,U_{12i+5},$$

$$(6.2)\quad A\sum_{j=1}^{4i} U_{3j}\,V_{3j+2} = A\sum_{j=1}^{4i} V_{3j}\,U_{3j+2} = U_{12i}\,V_{12i+5},$$

$$(6.3)\quad A\sum_{j=0}^{4i+1} U_{3j}\,U_{3j+2} = U_{12i+5}\,U_{12i+6},$$

$$(6.4)\quad A\sum_{j=0}^{4i+1} U_{3j}\,V_{3j+2} = A\sum_{j=0}^{4i+1} V_{3j}\,U_{3j+2} = V_{12i+5}\,U_{12i+6},$$

$$(6.5)\quad A\sum_{j=1}^{4i+2} U_{3j}\,U_{3j+2} = U_{12i+6}\,U_{12i+11},$$

$$(6.6)\quad A\sum_{j=1}^{4i+2} U_{3j}\,V_{3j+2} = A\sum_{j=1}^{4i+2} V_{3j}\,U_{3j+2} = U_{12i+6}\,V_{12i+11},$$

$$(6.7)\quad A\sum_{j=0}^{4i+3} U_{3j}\,U_{3j+2} = U_{12i+11}\,U_{12i+12},$$

$$(6.8)\quad A\sum_{j=0}^{4i+3} U_{3j}\,V_{3j+2} = A\sum_{j=0}^{4i+3} V_{3j}\,U_{3j+2} = V_{12i+11}\,U_{12i+12}.$$

The following formula is common to all three cases. Let $\alpha = 4i + u$.

Theorem 7. *For $(x, y) \in \{(0, 1), (1, 2), (2, 0)\}$, $u = 0, 1, 2, 3$ and for all integers $i \geq 0$, we have*

$$(7)\qquad \sum_{j=1}^{\alpha} V_{3j+x}\,V_{3j+y} = \Delta\sum_{j=1}^{\alpha} U_{3j+x}\,U_{3j+y}.$$

4. Sums of squares

In this section we consider summation of squares of numbers U_n and V_n. Of course, once again the indices of the summands are from the same residue class of the number 3. The summation formulas remain simple even when we multiply these squares with appropriate binomial coefficients. The formulae of the corresponding alternating sums are still unknown. Recall that $\alpha = 4i + u$ and $A = p\,T_3$.

Theorem 8. *(a) For $u = 0, 1, 2, 3$ and all integers $i \geq 0$, we have*

$$(8.1)\quad A\sum_{j=0}^{\alpha} U_{3j}^2 = U_{3\alpha}\,U_{3\alpha+3},$$

(b) For all integers $i \geq 0$, we have

$$(8.2)\quad A\sum_{j=0}^{\alpha} V_{3j}^2 = \begin{cases} \Delta\,U_{3\alpha}\,U_{3\alpha+3} + 4\,A, & \text{if } u = 0, 2; \\ \Delta\,U_{3\alpha}\,U_{3\alpha+3}, & \text{if } u = 1, 3. \end{cases}$$

Let $\lambda,\ \mu,\ \nu : \{0,1,2,3\} \to \{-1,0,1\}$ be defined by $\lambda(0) = -1$, $\lambda(1) = 0$, $\lambda(2) = 0$, $\lambda(3) = 1$, $\mu(0) = 1$, $\mu(1) = 0$, $\mu(2) = 0$, $\mu(3) = 1$, $\nu(0) = 0$, $\nu(1) = 0$, $\nu(2) = 1$ and $\nu(3) = 1$. Let $W^{(u)} = V$ for $u = 0$, 2 and $W^{(u)} = U$ for $u = 1$, 3. Similarly, let $Z^{(u)} = U$ for $u = 0$, 2 and $Z^{(u)} = V$ for $u = 1, 3$.

Theorem 9. *(a) For $u = 0,1,2,3$ and all integers $i \geq \mu(u)$, we have*

$$(9.1) \qquad \sum_{j=0}^{\alpha} \binom{\alpha}{j} U_{3j}^2 = \Delta^{2i+\lambda(u)}\, T_1^{\alpha}\, W_{3\alpha}^{(u)}.$$

(b) For $u = 0,1,2,3$ and $v = 0,1,2$ and all integers $i \geq \mu(u)$, we have

$$(9.2) \qquad \sum_{j=0}^{\alpha} \binom{\alpha}{j} V_{3j+v}^2 = \Delta \sum_{j=0}^{\alpha} \binom{\alpha}{j} U_{3j+v}^2.$$

Theorem 10. *For all integers $i \geq 0$, we have*

$$(10.1) \qquad A \sum_{j=0}^{\alpha} U_{3j+1}^2 = \begin{cases} U_{3\alpha+2}\, U_{3\alpha+3} + 2\,p, & \text{if } u = 0,2; \\ U_{3\alpha+2}\, U_{3\alpha+3}, & \text{if } u = 1,3. \end{cases}$$

$$(10.2) \qquad A \sum_{j=0}^{\alpha} V_{3j+1}^2 = \begin{cases} \Delta\, U_{3\alpha+2}\, U_{3\alpha+3} - 2\,p\,T_2, & \text{if } u = 0,2; \\ \Delta\, U_{3\alpha+2}\, U_{3\alpha+3}, & \text{if } u = 1,3. \end{cases}$$

Theorem 11. *For all integers $i \geq \mu(u)$, we have*

$$(11) \qquad \sum_{j=0}^{\alpha} \binom{\alpha}{j} U_{3j+1}^2 = \Delta^{2i+\lambda(u)}\, T_1^{\alpha}\, W_{3\alpha+2}^{(u)}.$$

Theorem 12. *For all integers $i \geq 0$, we have*

$$(12.1) \qquad A \sum_{j=0}^{\alpha} U_{3j+2}^2 = \begin{cases} U_{3\alpha+3}\, U_{3\alpha+4} - 2\,p, & \text{if } u = 0,2; \\ U_{3\alpha+3}\, U_{3\alpha+4}, & \text{if } u = 1,3. \end{cases}$$

$$(12.2) \qquad \sum_{j=0}^{\alpha} V_{3j+2}^2 = \begin{cases} \Delta\, U_{3\alpha+3}\, U_{3\alpha+4} + 2\,p\,T_2, & \text{if } u = 0,2; \\ \Delta\, U_{3\alpha+3}\, U_{3\alpha+4}, & \text{if } u = 1,3. \end{cases}$$

Theorem 13. *For all integers $i \geq \mu(u)$, we have*

$$(13) \qquad \sum_{j=0}^{\alpha} \binom{\alpha}{j} U_{3j+2}^2 = \Delta^{2i+\lambda(u)}\, T_1^{\alpha}\, W_{3\alpha+4}^{(u)}.$$

5. Sums of products with binomial coefficients

In this final section of the paper, we give formulae for sums from the Section 3 when the summands are multiplied with the appropriate binomial coefficient. In this case it is possible to get unified expressions so that binomial sums are much simpler.

For $X,\ Y \in \{U,\ V\}$, let $XY_{u,v,\alpha} = \sum_{j=0}^{\alpha} \binom{\alpha}{j} X_{3j+u}\, Y_{3j+v}$.

Theorem 14. *For* $(x,y) \in \{(0,1),(1,2),(2,0)\}$ *and for all integers* $u = 0$, 1, 2, 3 *and* $i \geq \mu(u)$, *we have*

$$UU_{x,y,\alpha} = \Delta^{2i+\lambda(u)}\, T_1^{\alpha}\, W^{(u)}_{3\alpha+x+y}, \tag{14.1}$$

$$VV_{x,y,\alpha} = \Delta\, UU_{x,y,\alpha}, \tag{14.2}$$

$$UV_{x,y,\alpha} = VU_{x,y,\alpha} = \Delta^{2i+\nu(u)}\, T_1^{\alpha}\, Z^{(u)}_{3\alpha+x+y}. \tag{14.3}$$

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