Applied Mathematical Sciences, Vol. 8, 2014, no. 165, 8279 - 8286
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2014.411915

A Laplace Type Problems for a Lattice with Cell Composed by Three Quadrilaterals and with Maximum Probability

D. Barilla, G. Caristi and E. Saitta

Department SEAM, University of Messina
Via dei Verid, 75, 98122 Messina, Italy

Dedicated to Professor Marius Stoka on the occasion of his 80^{th} birthday

Abstract

In some previous papers [1], [2], [3], [4], [5], [6], [7], [8], [9] and [10] the authors studies same Laplace problems with different fundamental cells. In this paper we consider a lattice with cell represented as in figure 1 and we compute the probability that a segment of random position and constant length intersects a side of lattice. Then we prove that there are values for parameters that determine the lattice and the length of segment for which the probability determined is maximum.

Mathematics Subject Classification: primary: 30C45, 30C80, secondary: 30D

Keywords: Geometric Probability, stochastic geometry, random sets, random convex sets and integral geometry

Let $\Re(a;\alpha)$ the lattice with the fundamental cell $C_0 = C_{01} \cup C_{02} \cup C_{03}$ represented in fig.1

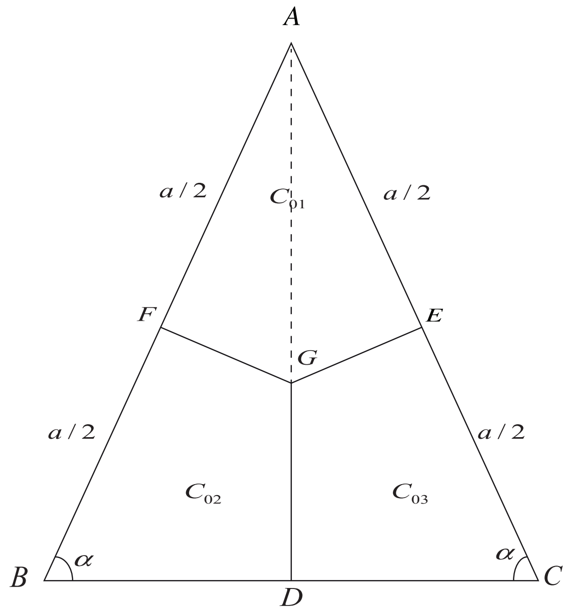

fig.1

where α is an angle with $\alpha\epsilon\,[\pi/4, \pi/3]$.

By this figure we have the following relations

$$|AD| = a\sin\alpha, \qquad |BD| = |CD| = a\cos\alpha, \qquad |BC| = 2a\cos\alpha,$$

$$|AG| = \frac{a}{2\sin\alpha}, \qquad |DG| = \frac{a\left(2\sin^2\alpha - 1\right)}{2\sin\alpha}, \qquad |FG| = |EG| = \frac{a}{2}ctg\alpha; \tag{1}$$

$$areaC_0 = \frac{a^2}{2}\sin 2\alpha. \tag{2}$$

We want to compute the probability that a segment s with random position and of constant length $l < \frac{a}{2}$ intersects a side of lattice, i.e. the probability P_{int} that s intersects a side of cell C_0.

The position of the segment s is determinated by middle point and by the angle φ that s forms with line BC.

To compute the probability P_{int} we consider the limit positions of segment s, for a fixed value of φ, situated in the cells C_{01}, C_{02}, C_{03}.

So, we have the fig. 2

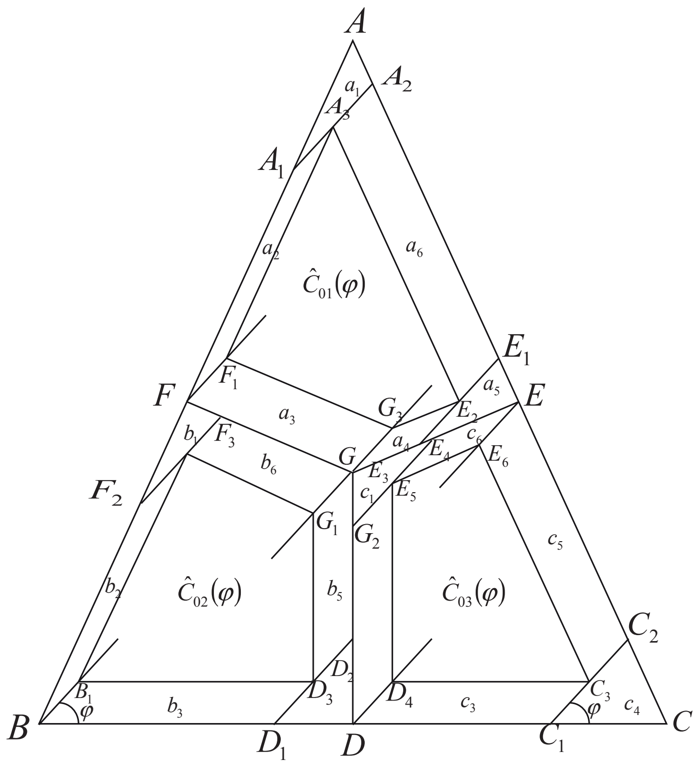

fig.2

and

$$area\widehat{C}_{01}\left(\varphi\right) = areaC_{01} - \sum_{i=1}^{6} areaa_i\left(\varphi\right), \tag{3}$$

$$area\widehat{C}_{02}\left(\varphi\right) = areaC_{02} - \sum_{i=1}^{6} areab_i\left(\varphi\right), \tag{4}$$

$$area\widehat{C}_{03}\left(\varphi\right) = areaC_{03} - \sum_{i=1}^{6} areac_i\left(\varphi\right). \tag{5}$$

Considering the fig. 2,

$$areaa_1\left(\varphi\right) = \frac{l^2 \sin\left(\alpha - \varphi\right)\sin\left(\varphi + \alpha\right)}{2\sin 2\alpha}, \tag{6}$$

$$areaa_2\left(\varphi\right) = \frac{al}{4}\sin\left(\alpha - \varphi\right) - \frac{l^2 \sin\left(\alpha - \varphi\right)\sin\left(\varphi + \alpha\right)}{2\sin 2\alpha}, \tag{7}$$

$$areaa_5\left(\varphi\right) = \frac{l^2}{4}\sin 2\left(\varphi + \alpha\right), \tag{8}$$

$$areaa_6(\varphi) = \frac{al}{4}\sin(\varphi+\alpha) - \frac{l^2}{4}\sin 2(\varphi+\alpha) - \frac{l^2\sin(\alpha-\varphi)\sin(\varphi+\alpha)}{2\sin 2\alpha}, \quad (9)$$

$$areaa_3(\varphi) = \frac{al}{4}ctg\alpha\cos(\alpha-\varphi), \quad (10)$$

$$areaa_4(\varphi) = \frac{l^2}{4}\sin 2(\varphi+\alpha) - \frac{al}{4}ctg\alpha\cos(\varphi+\alpha). \quad (11)$$

From relations (6), (7), (8), (9) (10) and (11) follows that

$$A_1(\varphi) = \sum_{i=1}^{6} areaa_i(\varphi) = \frac{al}{2}(\sin\alpha\cos\varphi + \cos\alpha\sin\varphi) +$$

$$\frac{l^2}{4}\sin 2\varphi - \frac{l^2}{4}.\frac{\cos 2\varphi + \cos 2\alpha}{\sin 2\alpha} \quad (12)$$

In the same way

$$areab_1(\varphi) = \frac{l^2}{4}\sin 2(\alpha-\varphi), \quad (13)$$

$$areab_2(\varphi) = \frac{al}{4}\sin(\alpha-\varphi) - \frac{l^2}{4}\sin 2(\alpha-\varphi), \quad (14)$$

$$areab_4(\varphi) = \frac{l^2}{4}\sin 2\varphi, \quad (15)$$

$$areab_3(\varphi) = \frac{al\cos\alpha}{2}\sin\varphi - \frac{l^2}{4}\sin 2\varphi, \quad (16)$$

$$areab_5(\varphi) = \frac{al\left(2\sin^2\alpha - 1\right)}{4\sin\alpha}\cos\varphi - \frac{l^2}{4}\sin 2\varphi, \quad (17)$$

$$areab_6(\varphi) = \frac{alctg\alpha}{4}\cos(\alpha-\varphi) - \frac{l^2}{4}\sin 2(\alpha-\varphi). \quad (18)$$

By relations (13), (14), (15) (16) (17) and (18) we get

$$A_2(\varphi) = \sum_{i=1}^{6} areab_i(\varphi) = al\cos\alpha\sin\varphi +$$

$$\frac{al}{2}\sin\alpha\cos\varphi - \frac{l^2}{4}\sin 2\varphi - \frac{l^2}{4}\sin 2(\alpha-\varphi) \quad (19)$$

At the end

$$areac_4(\varphi) = \frac{l^2 \sin\varphi \sin(\varphi+\alpha)}{2\sin\alpha}, \tag{20}$$

$$areac_3(\varphi) = \frac{al\cos\alpha}{2}\sin\varphi - \frac{l^2 \sin\varphi \sin(\varphi+\alpha)}{2\sin\alpha}, \tag{21}$$

$$areac_1(\varphi) = \frac{l^2 \cos\varphi \sin\left(\varphi+\alpha-\frac{\pi}{2}\right)}{2\sin\alpha}, \tag{22}$$

$$areac_2(\varphi) = \frac{al\left(2\sin^2\alpha - 1\right)}{4\sin\alpha}\cos\varphi - \frac{l^2 \cos\varphi \sin\left(\varphi+\alpha-\frac{\pi}{2}\right)}{2\sin\alpha}, \tag{23}$$

$$areac_5(\varphi) = \frac{al}{4}\sin(\varphi+\alpha) - \frac{l^2 \sin\varphi \sin(\varphi+\alpha)}{2\sin\alpha}, \tag{24}$$

$$areac_6(\varphi) = \frac{l^2 \cos\varphi \cos(\varphi+\alpha)}{2\sin\alpha} - \frac{al}{4}ctg\alpha\cos(\varphi+\alpha). \tag{25}$$

By formulas (20), (21), (22) (23) (24) and (25) we have that:

$$A_3(\varphi) = \sum_{i=1}^{6} areac_i(\varphi) = al\left(\sin\alpha - \frac{1}{2\sin\alpha}\right)$$

$$\cos\varphi + al\cos\alpha\sin\varphi - \frac{l^2}{2}\left(\sin 2\varphi - ctg\alpha\cos 2\varphi\right). \tag{26}$$

We denote with M_i $(i = 1, 2, 3)$ the set of segments s that they have the middle point in the cell C_{0i} and with N_i the set of segments s full content in C_{0i}, we have [12]:

$$P_{int} = 1 - \frac{\sum_{i=1}^{3}\mu(N_i)}{\sum_{i=1}^{3}\mu(M_i)}, \tag{27}$$

where μ is the Lebesgue measure in the Euclidean plane.

To compute the measure $\mu\ (M_i)$ and $\mu\ (N_i)$ we use the kinematic measure of Poincaré [11]:

$$dk = dx \wedge dy \wedge d\varphi,$$

where x, y are the coordinate of middle point of s and φ the fixed angle $\varphi\epsilon\,[0, \alpha]$ we have

$$\mu(M_i) = \int_0^{\alpha} d\varphi \int\int_{\{(x,y)\epsilon C_{0i}\}} dxdy =$$

$$\int_0^\alpha (areaC_{0i})\, d\varphi = \alpha areaC_{0i}, \qquad (i = 1, 2, 3).$$

then

$$\sum_{i=1}^{3} \mu(M_i) = \alpha \sum_{i=1}^{3} areaC_{0i} = \alpha areaC_0 = \frac{a^2}{2}\alpha \sin 2\alpha. \tag{28}$$

Moreover

$$\mu(N_i) = \int_0^\alpha d\varphi \int\int_{\left\{(x,y)\epsilon \widehat{C}_{0i}(\varphi)\right\}} dxdy = \int_0^\alpha \left[area\widehat{C}_{0i}(\varphi)\right] d\varphi =$$

$$\int_0^\alpha [areaC_{0i} - A_i(\varphi)]\, d\varphi = \alpha areaC_{0i} - \int_0^\alpha [A_i(\varphi)\, d\varphi], \qquad (i = 1, 2, 3),$$

then,

$$\sum_{i=1}^{3} \mu(N_i) = \frac{a^2}{2}\alpha \sin 2\alpha - \int_0^\alpha \left[\sum_{i=1}^{3} A_i(\varphi)\right] d\varphi. \tag{29}$$

The formulas (27), (28) and (29) give us

$$P_{int} = \frac{2}{a^2 \alpha \sin 2\alpha} \int_0^\alpha \left[\sum_{i=1}^{3} A_i(\varphi)\right] d\varphi. \tag{30}$$

We have that

$$\int_0^\alpha \left[\sum_{i=1}^{3} A_i(\varphi)\right] d\varphi = \frac{al}{2}\left(3 + 5\cos\alpha - 9\cos^2\alpha\right) - \frac{l^2}{8}\left(5 - 10\cos^2\alpha + 2\alpha ctg 2\alpha\right).$$

Replacing this relation in the (26) follows

$$P_{int} = \frac{1}{a^2 \alpha \sin 2\alpha}\left[al\left(3 + 5\cos\alpha - 9\cos^2\alpha\right) - \frac{l^2}{4}\left(5 - 10\cos^2\alpha + 2\alpha ctg 2\alpha\right)\right].$$

Considering

$$f(\alpha) = \frac{al\,(3 + 5\cos\alpha - 9\cos^2\alpha) - \frac{l^2}{4}\,(5 - 10\cos^2\alpha + 2\alpha ctg 2\alpha)}{\alpha \sin 2\alpha},$$

we have

$$f'(\alpha) = \frac{1}{\alpha^2 \sin^2 2\alpha} \left\{ \alpha \sin 2\alpha \left[al \sin\alpha \left(18\cos\alpha - 5\right) - \frac{l^2}{2}\left(5\sin 2\alpha + ctg2\alpha - \frac{2\alpha}{\sin^2 2\alpha}\right)\right] - \right.$$

$$\left. (\sin 2\alpha + 2\alpha\cos 2\alpha)\left[al\left(3 + 5\cos\alpha - 9\cos^2\alpha\right) - \frac{l^2}{4}\left(5 - 10\cos^2\alpha + 2\alpha ctg2\alpha\right)\right]\right\}.$$

If

$$\alpha \sin 2\alpha \left[al \sin\alpha \left(18\cos\alpha - 5\right) - \frac{l^2}{2}\left(5\sin 2\alpha + ctg2\alpha - \frac{2\alpha}{\sin^2 2\alpha}\right)\right] -$$

$$(\sin 2\alpha + 2\alpha\cos 2\alpha)\left[al\left(3 + 5\cos\alpha - 9\cos^2\alpha\right) - \frac{l^2}{4}\left(5 - 10\cos^2\alpha + 2\alpha ctg2\alpha\right)\right] = 0,$$

for $\alpha = \frac{\pi}{4}$ we have

$$f'(\alpha) = 0.$$

We prove that for these values we have

$$f''(\alpha) < 0,$$

then the probability

$$P_{int} = \frac{1}{2a^2} f(\alpha)$$

is maximum.

References

[1] D. Barilla, M. Bisaia, E. Saitta, M. Stoka, A Laplace type problems for a lattice with cell composed by two trapezium and a triangle, Applied Mathematical Sciences, Vol. 8, 2014, no. 9, 423 - 439.

[2] D. Barilla, M. Bisaia, G. Caristi and A. Puglisi, On Laplace type problems (I), Journal of Pure and Applied Mathematics: Advances and Applications, vol. 6, n.1 , 2011, pp. 51-70.

[3] D. Barilla, M. Bisaia, G. Caristi and A. Puglisi, On Laplace type problems (II), Far East Journal of Mathematical Sciences, vol. 58, n. 2, 2011, pp. 145-155.

[4] D. Barilla, G. Caristi, A. Puglisi, M. Stoka, A Laplace type problem for two hexagonal lattices of Delone with obstacles, Applied Mathematical Sciences, Vol. 7, no. 92, pp. 4571 - 4581.

[5] D. Barilla, G. Caristi, A. Puglisi, Laplace type for an irregular trapetium lattice and body test rectangle, Applied Mathematical Sciences vol. 7 n. 10 pp. 487-501.

[6] D. Barilla, G. Caristi, E. Saitta, M. Stoka, Laplace type problem for lattice with cell composed by two quadrilaterals and one triangle, Applied Mathematical Sciences, Vol. 8, 2014, no. 16, 789 - 804.

[7] D. Barilla, A. Puglisi, E. Saitta, M. Stoka, A Laplace type problems for a lattice with cell composed by two triangles and a trapezium, Applied Mathematical Sciences, Vol. 8, 2014, no. 9, 441 - 461.

[8] D. Barilla, A. Puglisi, E. Saitta, Laplace type problems for irregular lattice with cell "Pentagon + Triangle", International Mathematical Forum, Vol. 8, no. 23, pp. 1131 - 1142.

[9] G. Caristi, M. Stoka, A Laplace type problem for Delone Sessadecagonal lattice with obstacles, Mathematical Models and Methods in Applied Sciences Proceedings of the 13th WSEAS International Conference on Mathematics and Computers in Business and Economics (MCBE 12) pp. 124-130.

[10] G. Caristi, M. Stoka, Buffon-Laplace type problems for three regular lattices and "body test" a parallelogram, International Mathematical Forum, vol. 8 n. 6, pp. 245-271.

[11] H. Poincarè, Calcul des probabilitès, 2nd ed., Gauthier-Villard, Paris, 1912.

[12] M. Stoka, Probabilités géométriques de type "Buffon" dans le plan euclidien, Atti Accd. Sci. Torino, T, 110.

Received: September 1, 2014; Published: November 25, 2014