Detection and Estimation of Jumps in Discrete Time

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Dedicated to Professor Marius Stoka on the occasion of his 80th birthday

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Abstract

We consider n random functions defined on $[0, 1]$ and admitting a finite number of jumps. By using discrete observations we intend to detect the instants of jumps. As an Application, we obtain estimators of the intensity of jumps with an exponential rate.

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1 Introduction

Existence of jumps in random functions is very frequent: they appear in meteorology [cf 2], electricity consumption [cf 1, 7], finance [cf 9] and, more generally,

in the compound Poisson process. Another interesting example is the mistral gust [see 8].

The natural space for studying these jumps is the space $D = D[0, 1]$ of cadlag real functions defined over $[0, 1]$. Some facts concerning D will appear in Section 2.

Now, let X be a D -valued random variable admitting instants of jumps $t_1, \dots, t_k$ all distinct and belonging to $(0, 1)$, then, one may set

$$\Delta_j = |X(t_j) - X(t_j-)|,$$

with

$$0 < E\Delta_j < \infty, \quad 1 \leq j \leq k.$$

The problem is to detect $(t_j, j = 1, \dots, k)$ given i.i.d. copies of X_i observed at the instants $\frac{l}{q_n}, 0 \leq l \leq q_n, 1 \leq i \leq n$, where l and $q_n \geq 1$ are integers.

An Hölder regularity condition allows to detect $t_1, \dots, t_k$ at an exponential rate (cf Section 2).

Then, by using the detectors, one may again estimate $E\Delta_1, \dots, E\Delta_k$ at an exponential rate ; the main tool being the Bernstein's inequality.

Note that it is possible to replace the fixed instants by independent random instants. One may also extend the results to D -valued ARMA processes.

Concerning the proofs and the simulations, they will appear in Blanke and Bosq [5] ; see also Bosq [6] and Blanke and Bosq [4] for further results on this topic.

2 Detecting fixed jumps

Let D be the space of cadlag real functions defined over $[0, 1]$. Since the uniform norm $\|\cdot\|$ implies non separability, it is preferable to choose the Skorohod metric d [see 3, for details]. Thus, D will be equipped with the σ -algebra $\mathcal{D}$ associated with d .

Now, let X be a random variable which is D valued and $\mathcal{D}$ measurable. We suppose that X has k distinct instants of jumps $t_1, \dots, t_k$ belonging to $(0, 1)$ and with random intensity

$$\Delta_j = |X(t_j) - X(t_j-)|, \quad 1 \leq j \leq k.$$

We want to estimate $E\Delta_j$ from the i.i.d. copies $\left(X_i(\frac{l}{q_n}), 0 \leq l \leq q_n, 1 \leq i \leq n\right)$ where l and $q_n \geq 1$ are integers and $q_n \rightarrow \infty$ as $n \rightarrow \infty$.

Then, since the data are discrete, we need a regularity assumption. Below the t_j 's are classified in increasing order: $t_1^* < \dots < t_k^*$:

Assumption 2.1. For each $1 \leq i \leq n$:

$$|X_i(t) - X_i(s)| \leq M_i |t - s|^\alpha$$

where $(s, t) \in [0, t_1^*]^2 \cup \dots \cup [t_k^*, 1]^2$, $0 < \alpha \leq 1$, $EM_i^p < \infty$, $p \geq 1$.

Note that the fractional Brownian motion and the Ornstein-Uhlenbeck process with jumps satisfy Assumption 2.1 (with $n = 1$).

In order to detect the instants of jumps we consider the following approximation of t_j which is valid for q_n large enough:

$$0 < \frac{l_{jn} - 1}{q_n} < t_j \leq \frac{l_{jn}}{q_n} := t_{jn}, \quad 1 \leq j \leq k$$

also we set

$$\bar{\Delta}_{ln} = \frac{1}{n} \sum_{i=1}^n \left| X_i\left(\frac{l}{q_n}\right) - X_i\left(\frac{l-1}{q_n}\right) \right|, \quad 1 \leq l \leq q_n.$$

Finally we may and do suppose that

$$0 < E\Delta_k < \dots < E\Delta_1 < \infty,$$

and we consider the detectors

$$\hat{t}_{1n} = \frac{1}{q_n} \arg \max_{1 \leq l \leq q_n} \bar{\Delta}_{ln} := \frac{\hat{l}_{1n}}{q_n}$$

if $k \geq 2$ we put

$$\hat{t}_{2n} = \frac{1}{q_n} \arg \max_{1 \leq l \leq q_n, l \neq \hat{l}_{1n}} \bar{\Delta}_{ln} := \frac{\hat{l}_{2n}}{q_n}$$

and so on...

The following assumption implies that $\hat{t}_{1n}, \dots, \hat{t}_{kn}$ are almost surely unique:

Assumption 2.2. The distribution of $\bar{\Delta}_{ln}$ is continuous, $1 \leq l \leq q_n$.

Now, in order to obtain a convergence rate we suppose that we have:

Assumption 2.3. For some $c > 0$, $E(\exp(c\|X\|)) < \infty$.

Then, we get the following statement concerning the detector:

Proposition 2.4. If Assumption 2.1, 2.2 and 2.3 hold, then

$$P\left(\bigcup_{j=1}^k \{\hat{t}_{jn} \neq t_{jn}\}\right) = \mathcal{O}(\exp - nc_k), \quad c_k > 0.$$

Corollary 2.5. *Under the same assumptions, we obtain:*

$$P\left(\bigcup_{j=1}^k |\hat{t}_{jn} - t_j| > \frac{1}{q_n}\right) = \mathcal{O}(\exp -nc_k), \quad c_k > 0.$$

Consequently, almost surely for n large enough, the distance between $\hat{t}_{jn}$ and t_j is not greater than $\frac{1}{q_n}$. Note that this result cannot be improved since the rate of discretization is $\frac{1}{q_n}$.

3 Estimating the intensity of jumps

In order to estimate the intensity of jumps we use $\hat{t}_{jn}$ and set

$$\hat{D}_{jn} = \frac{1}{n} \sum_{i=1}^n \left| X_i(\hat{t}_{jn}) - X_i(\hat{t}_{jn} - \frac{1}{q_n}) \right|, \quad 1 \leq j \leq k.$$

First we get

Proposition 3.1. *If Assumption 2.2 and 2.3 hold, we have*

$$P\left(\hat{D}_{jn} - E \left| X(\hat{t}_{jn}) - X(\hat{t}_{jn} - \frac{1}{q_n}) \right| \geq \eta\right) = \mathcal{O}(\exp -nc(\eta)),$$

$1 \leq j \leq k, \quad (c(\eta) > 0).$

And finally

Corollary 3.2. *Under Assumption 2.1, 2.2 and 2.3, one obtains*

$$P\left(\left|\hat{D}_{jn} - E\Delta_j\right| \geq \eta\right) = \mathcal{O}(\exp -nd(\eta)),$$

$1 \leq j \leq k, \quad (d(\eta) > 0).$

Then, it follows that almost surely for n large enough, $\left|\hat{D}_{jn} - E\Delta_j\right| = \mathcal{O}\left(\sqrt{\frac{\log n}{n}}\right)$, $1 \leq j \leq k$. Some simulations show that the approximation of the intensities of jumps are satisfactory [cf 5].

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