A Buffon Type Problem for a Lattice with 
Fundamental Cell Composed by Two Triangles 
and Two Trapeziums

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Dedicated to Professor Marius Stoka on the occasion of his 80th birthday

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Abstract

In some previous papers [1], [2], [3], [4], [5], [6], [7], [8], [9] and 
[10] the authors studies same Buffon - Laplace problems with different 
fundamental cells. In this paper we compute the probability that a 
segment of constant length and random position intersects a side of 
lattice with the cell composed by two triangles and two trapeziums (fig. 1).

Mathematics Subject Classification: primary: 30C45, 30C80, secondary: 30D

Keywords: Geometric Probability, stochastic geometry, random sets, random convex sets and integral geometry

Let \( \mathbb{R} (a, b; \lambda; \alpha) \) the lattice with the fundamental cell

\[
C_0 = C_{01} \cup C_{02} \cup C_{03} \cup C_{04}
\]
where

\[0 < \lambda \leq \frac{1}{2}, \quad 0 < \beta \leq \alpha \leq \frac{\pi}{4}\].

By this figure are the follows relations

\[|AE| = |BF| = \frac{\lambda \alpha}{\cos \alpha}, \quad |CF| = |DE| = \frac{(1 - \lambda) \alpha}{\cos \beta},\]

\[|EF| = b - 2\lambda \alpha \tan \alpha,\]

\[\lambda \tan \alpha = (1 - \lambda) \tan \beta\] (1)

with

\[\lambda \tan \alpha = (1 - \lambda) \tan \beta\] (2)

All the same we have that

\[\text{area} C_{01} = (b - \lambda \alpha \tan \alpha) \lambda \alpha, \quad \text{area} C_{02} = (b - \lambda \alpha \tan \alpha)(1 - \lambda) \alpha,\]

\[\text{area} C_{03} = \text{area} C_{04} = \frac{\lambda \alpha^2}{2} \tan \alpha.\]

We want to compute the probability that a segment \(s\) of random position and constant length \(l < \min \left(\frac{b}{2}, \frac{\lambda \alpha}{2 \cos \alpha}\right)\) intersects a side of the lattice \(\mathbb{R}\), i.e. the probability \(P_{\text{int}}\) that \(s\) intersects a side of the fundamental cell \(C_0\).

The position of the segment \(s\) is determinated by middle point and by the angle that it formed with line \(BC\).

In order to compute \(P_{\text{int}}\) we consider the limit positions of segment \(s\), for a fixed value of \(\varphi\), in the cell \(C_{0i}\), \((i = 1, 2, 3, 4)\).

We have the fig. 2
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\[ a \quad b \quad \lambda a \quad (1-\lambda)a \]

\[ \begin{align*}
\text{area} \hat{C}_{01} (\varphi) &= \text{area} C_{01} - \sum_{i=1}^{6} \text{area} a_i (\varphi), \\
\text{area} \hat{C}_{02} (\varphi) &= \text{area} C_{02} - \sum_{i=1}^{6} \text{area} b_i (\varphi), \\
\text{area} \hat{C}_{03} (\varphi) &= \text{area} C_{03} - \sum_{i=1}^{5} \text{area} c_i (\varphi), \\
\text{area} \hat{C}_{04} (\varphi) &= \text{area} C_{04} - \sum_{i=1}^{5} \text{area} d_i (\varphi).
\end{align*} \]

Considering fig.1 we have

\[ \begin{align*}
\text{aread}_1 (\varphi) &= \frac{l^2 \sin \varphi \sin (\varphi + \beta)}{2 \sin \beta}, \\
\text{aread}_4 (\varphi) &= \frac{l^2 \sin \varphi \sin (\varphi - \alpha)}{2 \sin \alpha}, \\
\text{aread}_5 (\varphi) &= \frac{\alpha l}{2} \sin \varphi - \frac{l^2 \cot \alpha}{4\lambda} (1 - \cos 2\varphi), \\
\text{aread}_3 (\varphi) &= \left( \frac{\lambda \alpha}{\cos \alpha} - \frac{l \sin \varphi}{\sin \alpha} \right) \frac{l}{2} \sin (\varphi - \alpha), \\
\text{aread}_2 (\varphi) &= \left[ \frac{(1-\lambda) \alpha}{\cos \beta} - \frac{l \sin \varphi}{\sin \beta} \right] l \sin (\varphi + \beta).
\end{align*} \]

The formulas (7), (8), (9), (10), (11) and (12) give us
\[ \text{area}\hat{C}_{04}(\varphi) = \text{area}C_{04}(\varphi) - \]

\[
\frac{l}{2} \left[ \frac{(1 - \lambda)a}{\cos \beta} \sin(x + \beta) + \frac{\lambda a}{\cos \alpha} \sin(\varphi - \alpha) + \left(\alpha - \frac{l \tan \alpha}{\lambda}\right) \sin \varphi \right]. \tag{13}
\]

In the same way

\[ \text{area}_{a4}(\varphi) = \frac{l^2 \cos \varphi \sin(\varphi - \alpha)}{2 \cos \alpha}, \tag{14} \]

\[ \text{area}_{a3}(\varphi) = \left(\frac{\lambda \alpha}{\cos \alpha} - \frac{l \cos \varphi}{\cos \alpha}\right) \frac{l}{2} \sin(\varphi - \alpha), \tag{15} \]

\[ \text{area}_{a1}(\varphi) = \frac{l^2 \cos \varphi \sin(\varphi + \alpha)}{2 \cos \alpha}, \tag{16} \]

\[ \text{area}_{a5}(\varphi) = \left[ b - 2 \lambda \alpha \tan \alpha - \frac{l \sin(\varphi - \alpha)}{\cos \alpha}\right] \frac{l}{2} \cos \varphi, \tag{17} \]

\[ \text{area}_{a6}(\varphi) = \left(\frac{\lambda \alpha}{\cos \alpha} - \frac{l \cos \varphi}{\cos \alpha}\right) \frac{l}{2} \sin(\varphi + \alpha), \tag{18} \]

\[ \text{area}_{a2}(\varphi) = \left[ b - \frac{l \sin(\varphi + \alpha)}{\cos \alpha}\right] \frac{l}{2} \cos \varphi, \tag{19} \]

Replacing in the (4) the formulas (14), (15), (16), (17), (18) and (25) we have

\[ \text{area}\hat{C}_{01}(\varphi) = \text{area}C_{01} - \]

\[ \left[ \lambda al \sin \varphi + (b - \lambda \alpha \tan \alpha) l \cos \varphi - \frac{l^2}{2} \sin 2\varphi \right]. \tag{20} \]

Considering fig. 1

\[ \text{area}_{c1}(\varphi) = \frac{l^2 \sin \varphi \sin(\varphi + \alpha)}{2 \sin \alpha}, \tag{21} \]

\[ \text{area}_{c4}(\varphi) = \frac{l^2 \sin \varphi \sin(\varphi - \beta)}{2 \sin \beta}, \tag{22} \]

\[ \text{area}_{c5}(\varphi) = \left[ a - \frac{l \sin(\varphi - \alpha)}{\sin \alpha}\right] - \frac{l \sin(\varphi - \beta)}{\sin \beta} \frac{l}{2} \sin \varphi, \tag{23} \]

\[ \text{area}_{c2}(\varphi) = \left(\frac{\lambda \alpha}{\cos \alpha} - \frac{l \sin \varphi}{\sin \alpha}\right) \frac{l}{2} \sin(\varphi + \alpha), \tag{24} \]
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area_{C_3}(\varphi) = \left[ \frac{(1 - \lambda) a}{\cos \beta} - \frac{l \sin \varphi}{\sin \beta} \right] \frac{l}{2} \sin (\varphi - \beta). \quad (25)

Replacing in the (6) the relations (21), (22), (23), (24) and (25) we obtain

area\hat{C}_{03}(\varphi) = areaC_{03} - \left\{ \frac{al}{2} \sin \varphi + \left( \frac{\lambda a}{\cos \alpha} - \frac{l \sin \varphi}{\sin \alpha} \right) \right\} \frac{l}{2} \sin (\varphi - \beta). \quad (26)

In the same way

area_{b_1}(\varphi) = \frac{l^2 \cos \varphi \sin (\varphi - \beta)}{2 \cos \beta}, \quad (27)

area_{b_2}(\varphi) = \left[ b - 2\lambda \tan \alpha - \frac{l \sin (\varphi - \beta)}{\cos \beta} \right] \frac{l}{2} \cos \varphi, \quad (28)

area_{b_6}(\varphi) = \left[ \frac{(1 - \lambda) a}{\cos \beta} - \frac{l \cos \varphi}{\cos \beta} \right] \frac{l}{2} \sin (\varphi - \beta) - \frac{l^2 \cos \varphi \sin (\varphi - \beta)}{2 \cos \beta}, \quad (29)

area_{b_4}(\varphi) = \frac{l^2 \cos \varphi \sin (\varphi + \beta)}{2 \cos \beta}, \quad (30)

area_{b_3}(\varphi) = \left[ \frac{(1 - \lambda) a}{\cos \beta} - \frac{l \cos \varphi}{\cos \beta} \right] \frac{l}{2} \sin (\varphi + \beta), \quad (31)

area_{b_5}(\varphi) = \frac{bl}{2} \cos \varphi - \frac{l^2 \cos \varphi (\varphi + \beta)}{2 \cos \beta}. \quad (32)

Replacing in the (5) the relations (27), (28), (29), (30), (31) and (32) we have that

area\hat{C}_{02}(\varphi) = areaC_{02} \left[ (b - 2\lambda \tan \alpha) \frac{l}{2} \cos \varphi + \right.

\left( \frac{(1 - \lambda) a}{\cos \beta} \right) \frac{l}{2} \sin (\varphi + \beta) - \frac{bl}{2} \cos \varphi +

\left( \frac{(1 - \lambda) a}{\cos \beta} \right) \frac{l}{2} \sin (\varphi - \beta) - \frac{l^2}{2} \sin 2\varphi \right]. \quad (33)

Denoting with \(M_i\) \((i = 1, 2, 3, 4)\), the set of the segment \(s\) that have the middle point in the cell \(C_{01}\) and with \(N_i\) the set of the segment \(s\) completely in \(C_{0i}\) we have \([12]\)
\[ P_{\text{int}} = 1 - \frac{\sum_{i=1}^{4} \mu (N_i)}{\sum_{i=1}^{4} \mu (M_i)} \]  \hspace{1cm} (34)

where \( \mu \) is the Lebesgue measure in the euclidean plane.

To compute the measure \( \mu (M_i) \) and \( \mu (N_i) \) we use the kinematic measure of Poincaré [11]:

\[ dk = dx \wedge d\gamma \wedge d\varphi, \]

where \( x, y \) are the coordinate of middle point of segment \( s \) and \( \varphi \) the fixed angle.

We can write

\[ \mu (M_i) = \int_{\alpha}^{\pi/2} dy \int_{\{(x,y) \in C_{0i}\}} dx dy = \]

\[ \int_{0}^{\pi/2} (\text{area} C_{0i}) dy = \left( \frac{\pi}{2} - \alpha \right) C_{0i}, \quad (i = 1, 2, 3, 4). \]

then

\[ \sum_{i=1}^{4} \mu (M_i) = \left( \frac{\pi}{2} - \alpha \right) \text{area} C_{0}. \]  \hspace{1cm} (35)

All the same we have that

\[ \mu (N_i) = \int_{\alpha}^{\pi/2} d\varphi \int_{\{(x,y) \in C_{0i}(\varphi)\}} dx d\gamma = \int_{\alpha}^{\pi/2} \left[ \text{area} \hat{C}_{0i}(\varphi) \right] dy, \]

then

\[ \sum_{i=1}^{4} \mu (N_i) = \left( \frac{\pi}{2} - \alpha \right) \text{area} C_{0} - \]

\[ \{ \alpha l [(3 - \lambda) \cos \alpha - 2\lambda t g \alpha (1 - \sin \alpha)] + 2bl (1 - \cos \alpha) - \]

\[ \frac{l^2}{2} \left[ 1 + 2 \cos 2\alpha + \frac{1}{2(1 - \lambda)} \cot \alpha (\sin 2\alpha + \pi - 2\alpha) \right] \} . \]  \hspace{1cm} (36)

By formulas (53), (54) and (56) we have that

\[ P_{\text{int}} = \frac{2}{(\pi - 2\alpha)\alpha b} \{[(3 - \lambda) \cos \alpha - 2\lambda t g \alpha (1 - \sin \alpha)] \]
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\[ \alpha l + 2 (1 - \cos \alpha) bl - \frac{l^2}{4} \left[ 2 \cos 2\alpha + \frac{2 \cos^2 \alpha}{\lambda} + \frac{(\pi - 2\alpha) \cot \alpha}{\lambda} \right] \]. \quad (37)

In particular for \( \lambda = 1/2 \) and \( \alpha = \pi/4 \) the probability (57) can be write

\[ P = \frac{4}{\pi \alpha b} \left[ \left( \frac{7 \sqrt{2}}{4} - 1 \right) \alpha l + \left( 2 - \sqrt{2} \right) bl - \frac{l^2}{4} (2 + \pi) \right] . \]

At the end, for \( a = b \) this probability become

\[ P = \frac{3 \sqrt{2} + 4}{\pi} \frac{l}{\alpha} - \frac{2 + \pi}{\pi} \cdot \left( \frac{l}{2} \right)^2 . \]

References


Received: September 1, 2014; Published: November 25, 2014