A Laplace Type Problem for a Lattice of Rectangles with Triangular Obstacles

Uwe Bäsel* and Andrei Duma**

*HTWK Leipzig
Fakultät Maschinenbau und Energietechnik
04251 Leipzig, Germany

**FernUniversität in Hagen
Fakultät Mathematik und Informatik
58084 Hagen, Germany

Dedicated to Professor Marius Stoka on the occasion of his 80th birthday

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Abstract

In this paper, we calculate the probability that a small rectangle hits a lattice of rectangles with triangular obstacles.

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1 Introduction

A rectangle $\mathcal{D}$ with side lengths L and l , $L \geq l$, is randomly thrown onto a plane lattice $\mathcal{R}_{a,b,m}$ of rectangles with side lengths $4a$ and b , and $2a$ and b , respectively (see Fig. 1). In every corner of the rectangles there is an obstacle that is an isosceles right triangle with legs of length $m/2$ and hypotenuse of length $m/\sqrt{2}$ with $m < \min(2a, b)$.

Fig. 1: Lattice $\mathcal{R}_{a,b,m}$ with its fundamental cell $\mathcal{F}$

The fundamental cell $\mathcal{F}$ of $\mathcal{R}_{a,b,m}$ is shaded in Fig. 1. Using the Cartesian coordinate system in Fig. 2, we have

$$\mathcal{F} = \{(x, y) \in \mathbb{R}^2 \mid -2a \leq x \leq 2a, 0 \leq y \leq b/2\} \cup \{(x, y) \in \mathbb{R}^2 \mid -a \leq x \leq a, -b/2 \leq y \leq 0\}.$$

The area of $\mathcal{F}$ is equal to $3ab$. After attaching a fixed reference point C to $\mathcal{D}$, we define the *random throw of $\mathcal{D}$ onto $\mathcal{R}_{a,b,m}$* as follows: The coordinates x and y of C are random variables uniformly distributed in $[-2a, 2a]$, $[-b/2, b/2]$, respectively; the angle ϕ between the x -axis and the longer sides of $\mathcal{D}$ is a random variable uniformly distributed in $[0, 2\pi]$. All three random variables are stochastically independent. We only consider events with $C \in \mathcal{F}$.

In the following, we assume $\mathcal{D}$ to be small with respect to $\mathcal{R}_{a,b,m}$; this means

$$L \leq \min(2a - m, b - m, m/\sqrt{2}). \quad (1)$$

We denote by $d = \sqrt{L^2 + l^2}$ the length of the diagonals of $\mathcal{D}$, and by α the angle between a diagonal and the longer sides of $\mathcal{D}$, hence $\sin \alpha = l/d$ and $\cos \alpha = L/d$.

Known results for lattices with obstacles are for instance to be found in [1], [2], [3] and [4].

2 Hitting probability

Theorem 2.1. *If inequality (1) holds, then the probability p that $\mathcal{D}$ hits $\mathcal{R}_{a,b,m}$ is given by*

$$p = \frac{1}{6\pi ab} \left[4(3a+b)(L+l) - 4(L^2 + l^2) + \pi(L-l)^2 - 4(2 - \sqrt{2})m(L+l) + \pi m^2 \right]. \quad (2)$$

Proof. As reference point C of $\mathcal{D}$ we choose its barycentre. Due to the symmetry of $\mathcal{F}$ with respect to the y -axis, it is sufficient to consider ϕ only in the interval $[0, \pi/2]$ in order to get all positions of $\mathcal{D}$ exactly once. Let $\mathcal{F}(\mathcal{D}, \phi)$ be the set of all points $(x, y) \in \mathcal{F}$ where the rectangle $\mathcal{D}$ with coordinates (x, y) and angle ϕ does not intersect the lattice $\mathcal{R}_{a,b,m}$. Let $\mathcal{M}$ be the set of all rectangles $\mathcal{D}$ with barycentre in $\mathcal{F}$, and $\mathcal{N} \subset \mathcal{M}$ the set of all rectangles $\mathcal{D}$ with barycentre in $\mathcal{F}$, not intersecting $\mathcal{R}_{a,b,m}$. Then, the probability p is given by Stoka's formula

$$p = 1 - \frac{\mu(\mathcal{N})}{\mu(\mathcal{M})},$$

where μ is the Lebesgue measure in the Euclidean plane. We compute $\mu(\mathcal{M})$ and $\mu(\mathcal{N})$ by means of the elementary kinematic density $dx \wedge dy \wedge d\phi = dK$ for the group of motions in the plane (see [5, pp. 126-128], [6, p. 85]), and get

$$\begin{aligned} \mu(\mathcal{M}) &= \int_0^{\pi/2} d\phi \iint_{(x,y) \in \mathcal{F}} dx dy = \frac{\pi}{2} \text{area } \mathcal{F} = \frac{3\pi ab}{2}, \\ \mu(\mathcal{N}) &= \int_0^{\pi/2} d\phi \iint_{(x,y) \in \mathcal{F}(\mathcal{D}, \phi)} dx dy = \int_0^{\pi/2} \text{area } \mathcal{F}(\mathcal{D}, \phi) d\phi, \end{aligned}$$

hence

$$p = 1 - \frac{2}{3\pi ab} \int_0^{\pi/2} \text{area } \mathcal{F}(\mathcal{D}, \phi) d\phi. \quad (3)$$

Therefore, for the calculation of p it is sufficient to know the continuous function $[0, \pi/2] \rightarrow \mathbb{R}$, $\phi \mapsto \text{area } \mathcal{F}(\mathcal{D}, \phi)$, and to integrate it from 0 to $\pi/2$ with respect to ϕ . The set $\mathcal{F}(\mathcal{D}, \phi)$ consists of two hexagons. In order to calculate the dimensions of these hexagons, we have to distinguish the cases $\phi \in [0, \pi/4)$ and $\phi \in [\pi/4, \pi/2]$, see Figures 2 and 3.

In the case $\phi \in [0, \pi/4)$, $\mathcal{F}(\mathcal{D}, \phi)$ consists of

- a) two congruent trapezia $t(\phi)$ with parallel sides of lengths

$$\frac{b}{2} - \frac{d}{2} \sin(\alpha + \phi) \quad \text{and} \quad \frac{b}{2} - \frac{d}{2} \sin(\alpha + \phi) - \left(\frac{m}{2} - l \sin \phi \right),$$

and distance $(m/2) - l \sin \phi$,

b) two congruent trapezia $T(\phi)$ with parallel sides of lengths

$$\frac{b}{2} - \frac{d}{2} \sin(\alpha + \phi) \quad \text{and} \quad \frac{b}{2} - \frac{d}{2} \sin(\alpha + \phi) - \left(\frac{m}{2} - L \sin \phi \right),$$

and distance $(m/2) - L \sin \phi$,

c) one rectangle $\Delta_{4a}(\phi)$ with side lengths

$$4a - m - L(\cos \phi - \sin \phi) \quad \text{and} \quad \frac{b}{2} - \frac{d}{2} \sin(\alpha + \phi),$$

and one rectangle $\Delta_{2a}(\phi)$ with side lengths

$$2a - m - L(\cos \phi - \sin \phi) \quad \text{and} \quad \frac{b}{2} - \frac{d}{2} \sin(\alpha + \phi).$$

Therefore, we have

$$\begin{aligned} & \text{area } \mathcal{F}(\mathcal{D}, \phi) \\ &= 2 \cdot \frac{1}{2} \left[b - d \sin(\alpha + \phi) - \left(\frac{m}{2} - l \sin \phi \right) \right] \left(\frac{m}{2} - l \sin \phi \right) \\ & \quad + 2 \cdot \frac{1}{2} \left[b - d \sin(\alpha + \phi) - \left(\frac{m}{2} - L \sin \phi \right) \right] \left(\frac{m}{2} - L \sin \phi \right) \\ & \quad + [6a - 2m - 2L(\cos \phi - \sin \phi)] \left[\frac{b}{2} - \frac{d}{2} \sin(\alpha + \phi) \right] \\ &= 3ab - \frac{m^2}{2} - 3ad \sin(\alpha + \phi) - bl \sin \phi - bL \cos \phi + lm \sin \phi + Lm \sin \phi \\ & \quad + dl \sin(\alpha + \phi) \sin \phi + dL \sin(\alpha + \phi) \cos \phi - (l^2 + L^2) \sin^2 \phi, \end{aligned} \quad (4)$$

hence

$$\begin{aligned} & \int_0^{\pi/4} \text{area } \mathcal{F}(\mathcal{D}, \phi) \, d\phi \\ &= \frac{\pi}{4} \left(3ab - \frac{m^2}{2} \right) - 3 \left(\cos \alpha - \frac{1}{\sqrt{2}} \cos \alpha + \frac{1}{\sqrt{2}} \sin \alpha \right) ad \\ & \quad + \left(1 - \frac{\sqrt{2}}{2} \right) bl - \frac{\sqrt{2}}{2} bL + \left(1 - \frac{\sqrt{2}}{2} \right) lm + \left(1 - \frac{\sqrt{2}}{2} \right) Lm \\ & \quad + \left(\frac{\pi}{8} \cos \alpha - \frac{1}{4} \cos \alpha + \frac{1}{4} \sin \alpha \right) dl + \left(\frac{\pi}{8} \sin \alpha + \frac{1}{4} \sin \alpha + \frac{1}{4} \cos \alpha \right) dL \\ & \quad - \left(\frac{\pi}{8} - \frac{1}{4} \right) (l^2 + L^2). \end{aligned} \quad (5)$$

Fig. 2: Case $\phi \in [0, \pi/4]$

Fig. 3: Case $\phi \in [\pi/4, \pi/2]$

In the case $\phi \in [\pi/4, \pi/2]$, $\mathcal{F}(\mathcal{D}, \phi)$ is the union of the following polygons (see Fig. 3):

- a) two congruent trapezia $t^*(\phi)$ with parallel sides of lengths

$$\frac{b}{2} - \frac{d}{2} \sin(\alpha + \phi) \quad \text{and} \quad \frac{b}{2} - \frac{d}{2} \sin(\alpha + \phi) - \left(\frac{m}{2} - l \cos \phi \right),$$

and distance $(m/2) - l \cos \phi$,

- b) two congruent trapezia $T^*(\phi)$ with parallel sides of lengths

$$\frac{b}{2} - \frac{d}{2} \sin(\alpha + \phi) \quad \text{and} \quad \frac{b}{2} - \frac{d}{2} \sin(\alpha + \phi) - \left(\frac{m}{2} - l \cos \phi \right),$$

and distance $(m/2) - L \cos \phi$,

c) one rectangle $\delta_{4a}(\phi)$ with side lengths

$$4a - m - l(\sin \phi - \cos \phi) \quad \text{and} \quad \frac{b}{2} - \frac{d}{2} \sin(\alpha + \phi),$$

and one rectangle $\delta_{2a}(\phi)$ with side lengths

$$2a - m - l(\sin \phi - \cos \phi) \quad \text{and} \quad \frac{b}{2} - \frac{d}{2} \sin(\alpha + \phi).$$

Hence

$$\begin{aligned} & \text{area } \mathcal{F}(\mathcal{D}, \phi) \\ &= 2 \cdot \frac{1}{2} \left[b - d \sin(\alpha + \phi) - \left(\frac{m}{2} - l \cos \phi \right) \right] \left(\frac{m}{2} - l \cos \phi \right) \\ & \quad + 2 \cdot \frac{1}{2} \left[b - d \sin(\alpha + \phi) - \left(\frac{m}{2} - L \cos \phi \right) \right] \left(\frac{m}{2} - L \cos \phi \right) \\ & \quad + \left[6a - 2m - 2l(\sin \phi - \cos \phi) \right] \left[\frac{b}{2} - \frac{d}{2} \sin(\alpha + \phi) \right] \\ &= 3ab - \frac{m^2}{2} - 3ad \sin(\alpha + \phi) - bl \sin \phi - bL \cos \phi + lm \cos \phi + Lm \cos \phi \\ & \quad + dl \sin(\alpha + \phi) \sin \phi + dL \sin(\alpha + \phi) \cos \phi - (l^2 + L^2) \cos^2 \phi. \end{aligned} \quad (6)$$

As plausibility check, from (4) and (6) one finds that area $\mathcal{F}(\mathcal{D}, \cdot)$ is continuous in $\phi = \pi/4$ with

$$\begin{aligned} & \text{area } \mathcal{F}(\mathcal{D}, \pi/4) \\ &= 3ab - \frac{m^2}{2} - 3ad \sin \left(\alpha + \frac{\pi}{4} \right) - \frac{\sqrt{2}}{2} b(l + L) + \frac{\sqrt{2}}{2} m(l + L) \\ & \quad + \frac{\sqrt{2}}{2} d(l + L) \sin \left(\alpha + \frac{\pi}{4} \right) - \frac{1}{2} (l^2 + L^2). \end{aligned}$$

Now we get

$$\begin{aligned} & \int_{\pi/4}^{\pi/2} \text{area } \mathcal{F}(\mathcal{D}, \phi) d\phi \\ &= \frac{\pi}{4} \left(3ab - \frac{m^2}{2} \right) - 3 \left(\sin \alpha - \frac{1}{\sqrt{2}} \sin \alpha + \frac{1}{\sqrt{2}} \cos \alpha \right) ad \\ & \quad - \frac{\sqrt{2}}{2} bl - \left(1 - \frac{\sqrt{2}}{2} \right) bL + \left(1 - \frac{\sqrt{2}}{2} \right) lm + \left(1 - \frac{\sqrt{2}}{2} \right) Lm \\ & \quad + \left(\frac{\pi}{8} \cos \alpha + \frac{1}{4} \cos \alpha + \frac{1}{4} \sin \alpha \right) dl + \left(\frac{\pi}{8} \sin \alpha - \frac{1}{4} \sin \alpha + \frac{1}{4} \cos \alpha \right) dL \\ & \quad - \left(\frac{\pi}{8} - \frac{1}{4} \right) (l^2 + L^2). \end{aligned} \quad (7)$$

With $d \sin \alpha = l$ and $d \cos \alpha = L$, from (5) and (7) it follows that

$$\int_0^{\pi/2} \text{area } \mathcal{F}(\mathcal{D}, \phi) d\phi = \frac{\pi}{2} \left(3ab - \frac{m^2}{2} \right) - (3a + b)(L + l) \\ + \left(2 - \sqrt{2} \right) m(L + l) + L^2 + l^2 - \frac{\pi}{4} (L - l)^2. \quad (8)$$

The application of formula (3) completes the proof. $\square$

Remark 1. The probability p that a needle of length

$$L \leq \min \left(2a - m, b - m, \frac{m}{\sqrt{2}} \right)$$

hits $\mathcal{R}_{a,b,m}$ is given by

$$p = \frac{1}{6\pi ab} \left[4(3a + b)L - (4 - \pi)L^2 - 4 \left(2 - \sqrt{2} \right) mL + \pi m^2 \right].$$

Remark 2. From (2) and (8) it follows that in the set of all rectangles with constant perimeter $u = 2(l + L)$, the maximum of $\int_0^{\pi/2} \text{area } \mathcal{F}(\mathcal{D}, \phi) d\phi$ and hence the minimum of p is obtained only for squares with side length $u/4$.

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