Boundary Element Method of Modelling

Steady State Groundwater Flow

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Abstract

Water is a very important resource for any living organism and is mainly underground. The movement of groundwater is governed by the Laplace equation \( \nabla^2 h = 0 \), where \( h \) is the hydraulic head. Much research has been on groundwater modelling and many of them are inspired by Darcy’s law. The Boundary Element Method is one of the few widely used numerical techniques for solving boundary value problems in engineering and physical sciences. In this study, a steady state groundwater flow model has been analyzed using the boundary element method. The results show that water moves from points at higher hydraulic head to points at lower hydraulic head. This result using the boundary element method is in perfect agreement with the 2014 result by the same authors using the finite volume method.

Keywords: Groundwater Flow, Boundary Element Method, Green’s Identity, Discretization
1 Introduction

Groundwater models dictate how we translate flow systems for the calculation of groundwater flux and head [10]. Because of the simplifying assumptions embedded in the mathematical equations and the many uncertainties in the values of data required by the model, a model must be viewed as an approximation and not an exact duplication of field conditions. Groundwater models, however, even as approximations are a useful investigation tool. For the calculations one needs hydrological inputs, hydraulic parameters, initial and boundary conditions. The input is usually the inflow into the aquifer or the recharge, which varies temporally and spatially[11].

Groundwater flow and transport models simulate either steady state or transient flow. In steady-state systems, inputs and outputs are in equilibrium so that there is no net change in the system with time. In transient simulations, the inputs and outputs are not in equilibrium so there is a net change in the system with time. Steady state models provide average, long-term results. Transient models should be used when the groundwater regime varies over time[10]. Application of models to the analysis of steady flow in an aquifer requires knowledge of the spatial distributions of transitivity, boundary conditions and recharge rates with the aquifer[3]. In addition to the required knowledge in steady modeling, the transient model needs the initial conditions.

The general three-dimensional flow equation is given as:

$$\frac{\partial}{\partial x} \left( K_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( K_y \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z} \left( K_z \frac{\partial h}{\partial z} \right) = S_s \frac{\partial h}{\partial t}. \quad (1)$$

where $h$ is the hydraulic head, $K$ is the hydraulic conductivity and $S_s$ is the specific storage coefficient. If the hydraulic conductivities are assumed to be homogenous ($K_x, K_y, K_z$ are independent of $x, y, z$), the general equation can be written as:

$$K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} + K_z \frac{\partial^2 h}{\partial z^2} = S_s \frac{\partial h}{\partial t}. \quad (2)$$

When conductivity is also isotropic, that is $K_x = K_y = K_z = K$, this simplifies further to

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \nabla^2 h = \frac{S_s}{K} \frac{\partial h}{\partial t}, \quad (3)$$

where $\nabla^2 = (\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2})$ is called the Laplacian operator.
Equation (3) includes the storage term \( S_n \frac{\partial h}{\partial t} \) which occurs only in transient flow when \( \frac{\partial h}{\partial t} \neq 0 \) [8]. If however, the flow is steady state, \( \frac{\partial h}{\partial t} = 0 \), the right-hand-side of (3) become zero. Then we have:

\[
\nabla^2 h = 0.
\] (4)

Equation (4) is the well known Laplace equation and is the main concern in the present study since we model steady state flow.

2 Boundary Element Discretization

The boundary element method (BEM) is a powerful technique to solve boundary value problems numerically. The idea of BEM is that we can approximate the solution to a Partial Differential Equation (PDE) by looking at the solution to the PDE on the boundary and then use that information to find the solution in the interior of the domain [6]. The BEM is even more dominant for problems where the solution domain extends to infinity. Such problems occur frequently in acoustics [7]. The BEM also has wide applications in inverse problems [5].

In the present study, the boundary element method is employed to solve the two-dimensional Laplace Equation

\[
\nabla^2 u(r) = 0, \quad r = (x, y) \in \Omega \subset \mathbb{R}^2
\] (5)

in a domain \( \Omega \), with an enclosing boundary \( \Gamma \). On the boundary \( \Gamma \), either Dirichlet or Neumann boundary conditions are prescribed. Together, the governing partial differential equation within a domain and the boundary condition is called a Boundary Value Problem (BVP). The BVP can be classified as follows:

(1) Dirichlet B.V.P

\[
\nabla^2 u(r) = 0 \text{ in } \Omega ; \quad u(r) = \bar{u}(r) \text{ on } \Gamma;
\] (6)

(2) Neumann B.V.P

\[
\nabla^2 u(r) = 0 \text{ in } \Omega ; \quad \frac{\partial u(r)}{\partial n} = \bar{n}(r) \text{ on } \Gamma;
\] (7)
(3) Mixed B.V.P

\[ \nabla^2 u(r) = 0 \quad \text{in} \quad \Omega; \quad u(r) = \bar{u}(r) \quad \text{on} \quad \Gamma_1 \quad \text{and} \quad \frac{\partial u(r)}{\partial n} = \bar{u}_n(r) \quad \text{on} \quad \Gamma_2; \quad (8) \]

where

\[ \Gamma_1 \cup \Gamma_2 = \Gamma \quad \text{and} \quad \Gamma_1 \cap \Gamma_2 = \emptyset, \quad \bar{u}(r) \quad \text{and} \quad \bar{u}_n(r) \quad \text{are known functions defined on} \quad \text{the boundary.} \]

The solution to such a problem is principally the determination of \( u \) at points in the domain \( \Omega \), whether by analytic or numerical methods.

2.1 Fundamental solution

Before one applies the boundary element method to a particular problem, it is a must to obtain or know a fundamental solution for the problem. Fundamental solutions are tied to the Dirac Delta function [2]. Consider a point source placed at point \( p(x, y) \) of the xy-plane. Its density at \( r(\xi, \eta) \) may be expressed mathematically by the delta function as

\[ f(r) = \delta(r - p) \quad (9) \]

where

\[ \delta(r - p) = \begin{cases} 
1 & \text{when} \quad p \in \Omega, \\ 
0 & \text{when} \quad p \in \Omega^c \end{cases} \quad (10) \]

and the potential \( v = v(r, p) \) produced at point \( r \) satisfies the equation

\[ \nabla^2 v(r, p) = \delta(r - p). \quad (11) \]

A singular particular solution of equation (11) is called the fundamental solution of the potential equation [13]. It is determined by writing equation (11) in polar coordinates with origin at the point source \( p \). Since this solution is axisymmetric with respect to the source, it is independent of the polar angle \( \theta \).
The fundamental solution to the Laplace equation is
\[ v = \frac{1}{2\pi} \ln |r - p| = \frac{1}{2} \ln r \] (12)
where
\[ |r - p| = \sqrt{(\xi - x)^2 - (\eta - y)^2} = r \] (13)
and \( r(\xi, \eta) \) is any point in the domain and \( p(x, y) \) a point source. The fundamental solution (12) is also known as the free space Green’s function for Laplace equation. The fundamental solution of a particular equation is the weighting function that is used in the boundary element formulation of that equation [2].

2.2 Integral Formulation

To develop the boundary element method for the solution of Laplace equation in a two-dimensional domain, we first form an integral equation from the Laplace equation by using a weighted integral equation and then use the Green’s second identity [4].

**Theorem 2.1 (Green’s Second Identity).** Consider the functions \( u = u(x, y) \) and \( v = v(x, y) \) which are twice continuously differentiable in the domain \( \Omega \) and once on the boundary \( \Gamma \). Then,
\[
\int_{\Omega} (v \nabla^2 u - u \nabla^2 v) d\Omega = \int_{\Gamma} (v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}) d\Gamma,
\] (14)
where
\[
\frac{\partial}{\partial n} \equiv n \nabla = (n_x i + n_y j) \cdot (i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y}) = n_x \frac{\partial}{\partial x} + n_y \frac{\partial}{\partial y}
\]
is the operator that produces the derivative of a function in the direction of \( n \). \( n \)

Applying Green’s identity (14), for the function \( u \) and \( v \) that satisfy equations (5) and (11), respectively, and assuming that the source lies at point \( p \in \Omega \), we obtain

\[
- \int_{\Omega} u(r) \delta(r - p) d\Omega_r = \int_{\Gamma} \left[ v(\hat{r}, p) \frac{\partial u(\hat{r})}{\partial n_{\hat{r}}} - u(\hat{r}) \frac{\partial v(\hat{r}, p)}{\partial n_{\hat{r}}} \right] ds_{\hat{r}}.
\]

(15)

where \( p \) is a fixed point in \( \Omega \), \( r \) and \( \hat{r} \) are variable points in \( \Omega \) and in \( \Gamma \) respectively.

Since \( v(r, p) = v(p, r) \) by equation (12), equation (15) is written as

\[
- \int_{\Omega} u(p) \delta(p - r) d\Omega_r = \int_{\Gamma} \left[ v(p, \hat{r}) \frac{\partial u(\hat{r})}{\partial n_{\hat{r}}} - u(p) \frac{\partial v(p, \hat{r})}{\partial n_{\hat{r}}} \right] ds_{\hat{r}},
\]

(16)

where \( \delta(r - p) \) is the Dirac delta function in two dimension defined as

\[
\int_{\Omega} \delta(p - r) h(p) d\Omega_p = h(r), \quad p(x, y), r(\xi, \eta) \in \Omega
\]

(17)

for an arbitrary function \( h(p) \), which is continuous in the domain \( \Omega \) containing the point \( r(\xi, \eta) \).

The two-dimensional delta function may also be described by

\[
\delta(p - r) = \begin{cases} 
0, & p \neq r, \\
\infty, & p = r
\end{cases}
\]

(18)

and

\[
\int_{\Omega} \delta(p - r) d\Omega_p = \int_{\Omega^*} \delta(p - r) d\Omega_p = 1, \quad r(\xi, \eta) \in \Omega^* \subseteq \Omega.
\]

(19)

2.3 Integral equation for points inside \( \Omega \)

Since the fundamental solution \( v \) is singular at the point \( p \) when \( p = r \), the domain of integration where Green’s second identity is applied must be defined isolating the point \( p \) [1]. We construct the ball \( \Omega_\epsilon \) of radius \( \epsilon \) around it as shown in Figure 2. Then the new domain of integration is \( \Omega - \Omega_\epsilon \) with boundary \( \Gamma + \Gamma_\epsilon \).
Applying equation (16) on $\Omega - \Omega_\epsilon$ and taking the limit as $\epsilon \to 0$, we have

$$\lim_{\epsilon \to 0} \int_{\Omega - \Omega_\epsilon} u(p) \delta(p-r)d\Omega_r = - \int_{\Gamma} \left[ v(p, \hat{r}) \frac{\partial u(\hat{r})}{\partial n_{\hat{r}}} - u(p) \frac{\partial v(p, \hat{r})}{\partial n_{\hat{r}}} \right] ds_{\hat{r}}. \quad (20)$$

Since $p$ is a fixed point, equation (20) can be written as:

$$u(p) \lim_{\epsilon \to 0} \int_{\Omega - \Omega_\epsilon} \delta(p-r)d\Omega_r = - \int_{\Gamma} \left[ v(p, \hat{r}) \frac{\partial u(\hat{r})}{\partial n_{\hat{r}}} - u(p) \frac{\partial v(p, \hat{r})}{\partial n_{\hat{r}}} \right] ds_{\hat{r}}. \quad (21)$$

By the property (19) of the Dirac Delta function, we get

$$u(p) = - \int_{\Gamma} \left[ v(p, \hat{r}) \frac{\partial u(\hat{r})}{\partial n_{\hat{r}}} - u(p) \frac{\partial v(p, \hat{r})}{\partial n_{\hat{r}}} \right] ds_{\hat{r}}. \quad (22)$$

The functions $v$ and $\frac{\partial v}{\partial n}$ in the foregoing equation are both known quantities. These are the fundamental solution of the Laplace equation and its normal derivative $\nabla v \cdot n$, which is given by

$$\frac{\partial v}{\partial n} = \frac{1}{2\pi} \frac{\cos \phi}{r}, \quad (23)$$

where $r = |r - p|$ and $\phi = \text{angle}(r, n)$ (see Figure 1).

The expression (22) is the solution of the Laplace equation at any point $p$ inside the domain $\Omega$ (not on the boundary $\Gamma$) in terms of the boundary values of $u$ and its normal derivative $\frac{\partial u}{\partial n}$. The expression (22) is called the integral representation of the solution for the Laplace equation. It is apparent from
the boundary conditions (6) and (7), that only one of the quantities \( u \) or \( \frac{\partial u}{\partial n} \) is prescribed at a point \( \mathbf{r}(\xi, \eta) \) on the boundary. Consequently, it is not yet possible to determine the solution from the integral representation (22). For this purpose, we evaluate the boundary quantity which is not yet prescribed by the boundary conditions (either \( u \) or \( \frac{\partial u}{\partial n} \)), by deriving the integral representation of \( u \) for points \( \mathbf{p} \) lying on the boundary \( \Gamma \).

### 2.4 Integral equation for points on the boundary \( \Gamma \)

We study the general case where the boundary is not smooth and \( \mathbf{p} \) is a corner point where the fundamental solution \( \mathbf{v} \) is singular. We consider the domain \( \Omega^* \) which results from \( \Omega \) after subtracting a small circular section with center \( \mathbf{p} \), radius \( \epsilon \) and confined by the arcs \( \mathbf{Ap} \) and \( \mathbf{pB} \). The circular arc \( \mathbf{AB} \) is denoted \( \Gamma_\epsilon \) and the sum of the arcs \( \mathbf{Ap} \) and \( \mathbf{pB} \) by \( l \). The outward normal to \( \Gamma_\epsilon \) coincides with the radius \( \epsilon \) and is directed towards the center \( \mathbf{p} \). The angle between the tangents of the boundary at point \( \mathbf{p} \) is denoted by \( \alpha \). Hence, it is obvious that,

\[
\lim_{\epsilon \to 0} (\theta_1 - \theta_2) = \alpha,
\]

\[
\lim_{\epsilon \to 0} \Gamma_\epsilon = 0,
\]

and

\[
\lim_{\epsilon \to 0} (\Gamma - l) = \Gamma.
\]

![Figure 3: Geometric definitions related to a corner point \( \mathbf{p} \) of a non-smooth boundary [4].](image)

Next we apply Green’s identity in the domain \( \Omega^* \) for the functions \( u \) and \( v \) satisfying equations (5) and (11) respectively. Since point \( \mathbf{p} \) lies outside the
domain $\Omega^*$, where $\delta(\mathbf{r} - \mathbf{p}) = 0$, it follows that
\[\int_{\Omega^*} u(\mathbf{p})\delta(Q - P)d\Omega = 0 \quad (27)\]
and consequently Green’s identity generates
\[0 = \int_{\Gamma_{-t}} \left( v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds + \int_{\Gamma} \left( v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds = I_1 + I_2. \quad (28)\]

We will examine next the behavior of the integrals in the above equation when $\epsilon \to 0$. The integral $I_1$ becomes
\[I_1 = \lim_{\epsilon \to 0} \int_{\Gamma_{-t}} \left( v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds = \int_{\Gamma} \left( v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds. \quad (29)\]
The integral $I_2$ is written as
\[I_2 = \int_{\Gamma_t} \left( v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds = \int_{\Gamma_t} \frac{1}{2\pi} \frac{\partial u}{\partial n} \ln rs - \int_{\Gamma_t} \frac{1}{2\pi} \frac{u \cos \phi}{r} ds = \hat{I}_1 + \hat{I}_2. \quad (30)\]

For the circular arc $\Gamma$, its $r = \epsilon$ and $\phi = \pi$. Moreover, $ds = \epsilon(-d\theta)$, because the angle $\theta$ in the counter-clockwise sense, which is opposite to that of increasing $s$. Therefore, the first of the resulting two integrals in equation (30), takes the form
\[\hat{I}_1 = \int_{\Gamma_t} \frac{1}{2\pi} \frac{\partial u}{\partial n} \ln rs = \int_{\theta_1}^{\theta_2} \frac{1}{2\pi} \frac{\partial u}{\partial n} \epsilon \ln \epsilon d(-\theta). \quad (31)\]

By the mean value theorem of integral calculus, the value of the integral is equal to the value of integrand at some point $O$ within the integration interval multiplied by the length of that interval. Hence,
\[\hat{I}_1 = \frac{1}{2\pi} \left[ \frac{\partial u}{\partial n} \right]_O \epsilon \ln \epsilon(\theta_1 - \theta_2). \quad (32)\]

When $\epsilon \to 0$, the point $O$ of the arc approaches point $\mathbf{p}$. In this case, the derivative $[\frac{\partial u}{\partial n}]_\mathbf{p}$, though not defined, is bounded. Nevertheless, it is
\[\lim_{\epsilon \to 0} (\epsilon \ln \epsilon) = 0\]
which implies that
\[\lim_{\epsilon \to 0} \hat{I}_1 = 0 \quad (33)\]

In a similar way, the second integral $\hat{I}_2$ of equation (30) may be written as
\[\hat{I}_2 = -\int_{\Gamma_t} \frac{1}{2\pi} \frac{u \cos \phi}{r} ds = -\int_{\theta_1}^{\theta_2} \frac{1}{2\pi} \frac{1}{\epsilon} u(-\epsilon \cos \theta) \quad (34)\]
by applying the mean value theorem
\[
\dot{I}_2 = -\frac{1}{2\pi} u_0(\theta_2 - \theta_1) = \frac{\theta_1 - \theta_2}{2\pi} u_0
\]
and finally, by condition (24)
\[
\lim_{\epsilon \to 0} I_2 = \frac{\alpha}{2\pi} u(p). \tag{36}
\]
By virtue of equations (33) and (36), (30) yields
\[
\lim_{\epsilon \to 0} \int_{\Gamma} \left[ v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right] ds = \frac{\alpha}{2\pi} u(p). \tag{37}
\]
Incorporating the findings of equations (29) and (37) into (28), the latter gives for \( \epsilon \to 0 \),
\[
\frac{\alpha}{2\pi} u(p) = -\int_{\Gamma} \left[ v(p, \hat{r}) \frac{\partial u(\hat{r})}{\partial n} - u(\hat{r}) \frac{\partial v(p, \hat{r})}{\partial n} \right] d\hat{s}, \tag{38}
\]
which is the integral representation of the solution for the Laplace equation at points \( p \in \Gamma \), where the boundary is not smooth. For point \( p \), where the boundary is smooth, its \( \alpha = \pi \) and thus, equation (38) becomes
\[
\frac{1}{2} u(p) = -\int_{\Gamma} \left[ v(p, \hat{r}) \frac{\partial u(\hat{r})}{\partial n} - u(\hat{r}) \frac{\partial v(p, \hat{r})}{\partial n} \right] d\hat{s}. \tag{39}
\]
A comparison between equations (22) and (38) reveals that the function \( u \) is discontinuous when the point \( p \in \Omega \) approaches point \( \hat{p} \in \Gamma \). It exhibits a jump equal to \( (1 - \frac{\alpha}{2\pi}) \) for corner points, or \( \frac{1}{2} \) for points on smooth parts of the boundary \( \Gamma \).

### 2.5 Integral equation for points outside the domain \( \Omega \)

When the point \( p \) is located outside the domain \( \Omega \), Green’s identity gives
\[
0 = -\int_{\Gamma} \left[ v(p, \hat{r}) \frac{\partial u(\hat{r})}{\partial n} - u(\hat{r}) \frac{\partial v(p, \hat{r})}{\partial n} \right] d\hat{s}. \tag{40}
\]
Equations (22), (39) and (40) can be combined in a single general equation as
\[
\epsilon(p) u(p) = -\int_{\Gamma} \left[ v(p, \hat{r}) \frac{\partial u(\hat{r})}{\partial n} - u(\hat{r}) \frac{\partial v(p, \hat{r})}{\partial n} \right] d\hat{s}, \tag{41}
\]
where \( \epsilon(p) \) is a coefficient which depends on the position of point \( p \) [13] and is defined as

\[
\epsilon(p) = \begin{cases} 
1 & \text{for } p \text{ inside } \Omega, \\
\frac{1}{2} & \text{for } p \text{ on a smooth boundary } \Gamma, \\
0 & \text{for } p \text{ outside } \Omega.
\end{cases}
\]

Equation (39) constitutes a compatibility relation between the boundary values of \( u \) and \( \frac{\partial u}{\partial n} \), meaning that only one of the quantities \( u \) and \( \frac{\partial u}{\partial n} \) can be prescribed at each point of the boundary. At the same time, equation (39) can be viewed as an integral equation on the boundary \( \Gamma \), that is, a boundary integral equation with unknown as the quantity which is not prescribed by the boundary condition. In the following development of BEM, we assume a smooth boundary \( \Gamma \). Thus, for the Dirichlet problem with \( u = \bar{u} \) on \( \Gamma \), equation (39) is written as

\[
\frac{1}{2} \bar{u} = - \int_{\Gamma} \left( v \frac{\partial u}{\partial n} - \bar{u} \frac{\partial v}{\partial n} \right) ds \tag{42}
\]

in which the only unknown is the function \( \frac{\partial u}{\partial n} \) on \( \Gamma \). For the Neumann problem, \( \frac{\partial u}{\partial n} = \bar{n} \), equation (39) becomes

\[
\frac{1}{2} u = - \int_{\Gamma} \left( v \bar{n} - u \frac{\partial v}{\partial n} \right) ds \tag{43}
\]

with only the unknown as the function \( u \) on \( \Gamma \). For problems with mixed boundary conditions, equation (39) is treated as two separate equations, namely

\[
\frac{1}{2} \bar{u} = - \int_{\Gamma_1} \left( v \frac{\partial u}{\partial n} - \bar{u} \frac{\partial v}{\partial n} \right) ds \quad \text{on } \Gamma_1 \tag{44}
\]

and

\[
\frac{1}{2} u = - \int_{\Gamma_2} \left( v \bar{n} - u \frac{\partial u}{\partial n} \right) ds \quad \text{on } \Gamma_2. \tag{45}
\]

### 2.6 Undergroundwater Flow Model For the Present Study

The flow area is assumed rectangular with Neumann boundary conditions on two sides and Dirichlet boundary conditions on the other two as shown by Figure 4. We consider isotropic flow which leads to the Laplace equation.
3 Implementation of BEM

We now show how equation (41) may be applied to obtain a simple boundary element procedure for solving numerically the interior boundary value problem defined by equations (6), (7) and (8). The advantage of the BEM is to discretize the boundary into finite number of segments, not necessarily equal, which are called boundary elements or elements. The usually employed boundary elements are the constant elements, the linear elements and the parabolic or quadratic elements[4]. On each element, we distinguish the extreme points or end points and the nodes or nodal points. The latter are the points at which values of the boundary quantities are assigned. In the case of constant elements, the boundary segment is approximated by a straight line, which connects its extreme points. The node is placed at the mid-point of the straight line and the boundary quantity is assumed to be constant along the element and equal to its value at the nodal point as shown in Figure 5 [12]. In this work, the numerical solution of the integral equation (41) will be presented using constant boundary elements.
3.1 The BEM with constant boundary elements

The boundary $\Gamma$ is approximated as a union of elements. That is, $\bigcup_{j=1}^{N} \Gamma_j \simeq \Gamma$. The values of the boundary quantity $u$ and its normal derivative $\frac{\partial u}{\partial n}$ are assumed constant over each element and equal to their value at the mid-point of the element.

![Figure 5: Discretisation of a circle into elements $\Gamma_i$](image)

The discretized form of equation \((39)\) is expressed for a given point $p_i$ on $\Gamma$ as

$$\frac{1}{2} u^i = - \sum_{j=1}^{N} \int_{\Gamma_j} v(p_i, \mathbf{r}) \frac{\partial u(\mathbf{r})}{\partial n_\mathbf{r}} d\xi + \sum_{j=1}^{N} \int_{\Gamma_j} u(\mathbf{r}) \frac{\partial v(p_i, \mathbf{r})}{\partial n_\mathbf{r}} d\xi \tag{46}$$

where $\Gamma_j$ is the element on which $j$-th node is located and over which integration is carried out, $p_i$ is the nodal point of the $i$-th element and $u^i$ is the value of the function $u$ at the point $p_i$. For constant elements, the boundary is smooth at the nodal points, hence $\epsilon(p) = \frac{1}{2}$. The values of $u$ and $\frac{\partial u}{\partial n}$ denoted $u_n$ can be moved outside the integral since they are constant on each element. Denoting $u$ by $u^j$ and $u_n$ by $u_n^j$ on the $j$-th element, equation \((46)\) may be written as

$$-\frac{1}{2} u^i + \sum_{j=1}^{N} \left( \int_{\Gamma_j} \frac{\partial v}{\partial n} ds \right) u^j = \sum_{j=1}^{N} \left( \int_{\Gamma_j} v ds \right) u_n^j. \tag{47}$$

The values of the foregoing integrals express the contribution of the nodal values $u^j$ and $u_n^j$ to the formation of the value $\frac{1}{2} u^i$. For this reason, they are often referred to as influence coefficients \([4]\). These coefficients are denoted...
and defined by
\[ \tilde{H}_{ij} = \int_{\Gamma_j} \frac{\partial v(p_i, \mathbf{r})}{\partial n_{\xi}} ds_{\xi} \quad \text{and} \quad G_{ij} = \int_{\Gamma_j} v(p_i, \mathbf{r}) ds \] (48)

where the point \( p_i \) remains fixed (reference point), while the point \( \mathbf{r} \) varies over the \( j \)-th element (integration point) (Figure:6).

Introducing the notations (48) into equation (47) the discrete form of the solution becomes
\[ -\frac{1}{2} u^i + \sum_{j=1}^{N} \tilde{H}_{ij} u^j = \sum_{j=1}^{N} G_{ij} u^n_j \] (49)

Moreover, setting
\[ H_{ij} = \tilde{H}_{ij} - \frac{1}{2} \delta_{ij} \] (50)

where \( \delta_{ij} \) is the Kronecker Delta which is defined as \( \delta_{ij} = 0 \) for \( i \neq j \) and \( \delta_{ij} = 1 \) for \( i = j \), equation (46) may further be written as
\[ \sum_{j=1}^{N} H_{ij} u^j = \sum_{j=1}^{N} G_{ij} u^n_j \] (51)

Equation (51) is applied consecutively for all the nodes \( p_i \) \( (i = 1, 2, ..., N) \) yielding a system of \( N \) linear algebraic equations, which are arranged in the matrix form as;
\[ \mathbf{H} \mathbf{u} = \mathbf{G} \mathbf{u}_n. \] (52)

The matrix \( \mathbf{H} \) given by equation (50) and \( \mathbf{G} \) by equation (48) are \( N \times N \) square matrices and \( \mathbf{u} \) and \( \mathbf{u}_n \) are vectors of dimension \( N \).
Assuming mixed boundary conditions, in this case, the part $\Gamma_1$ of the boundary on which $u$ is prescribed and the part $\Gamma_2$ on which $u_n$ is prescribed, are discretized into $N_1$ and $N_2$ constant elements, respectively ($\Gamma_1 \cup \Gamma_2 = \Gamma$, $N_1 + N_2 = N$). Hence, equation (52) again contains $N$ unknowns, that is $N - N_1$ values of $u$ on $\Gamma_2$ and $N - N_2$ values of $u_n$ on $\Gamma_1$. These $N$ unknowns may be determined from the system of equations (52).

We have to separate the unknown from the known quantities when we solve the system. By partitioning the matrices, equation (52) can be written as

\[
\begin{bmatrix}
H_{11} & H_{12} \\
\end{bmatrix}
\begin{bmatrix}
\{\overline{u}\}_1 \\
\{u\}_2 \\
\end{bmatrix} =
\begin{bmatrix}
G_{11} & G_{12} \\
\end{bmatrix}
\begin{bmatrix}
\{u_n\}_1 \\
\{\overline{u_n}\}_2 \\
\end{bmatrix}
\] (53)

where $\{\overline{u}\}_1$ and $\{\overline{u_n}\}_2$ are the known quantities on $\Gamma_1$ and $\Gamma_2$ respectively, while $\{u_n\}_1$ and $\{u\}_2$ denote the corresponding unknown ones. Taking all the unknown to the left hand side of the equation and the known ones to the right hand side, we obtained

\[
Ax = b
\] (54)

where

\[
A = [H_{12} - G_{11}],
\] (55)

\[
x = \begin{bmatrix}
\{u\}_2 \\
\{u_n\}_1 \\
\end{bmatrix}
\] (56)

\[
b = -H_{11}\{\overline{u}\}_1 + G_{12}\{\overline{u_n}\}_2.
\] (57)

$A$ is an $N \times N$ square matrix, and $x$, $b$ vectors with dimensions $N[12]$.

Matrices $A$ and $b$ can also be constructed using an alternative straightforward procedure, which is based on the observation that matrix $A$ will eventually contain all the columns of $H$ and $G$ that correspond to the unknown boundary values of $u$ and $u_n$, whereas vector $b$ will result as the sum of those columns of $H$ and $G$, which correspond to the known values $u$ and $u_n$, after they have been multiplied by these values. It should be noted that a change of sign occurs, when the columns of $G$ or $H$ are moved to the other side of the equation [4].

The solution of the simultaneous equations (54) yield the unknown boundary quantities $u$ and $u_n$. Therefore, knowing all the boundary quantities on $\Gamma$, the solution $u$ can be computed at any point $p(x, y)$ in the domain $\Omega$ by virtue of
equation (41) for $\epsilon(p) = 1$. Applying the same discretization as in equation (46), we arrived at the following expression

$$u(p) = \sum_{j=1}^{N} \hat{H}_{ij} u^j - \sum_{j=1}^{N} G_{ij} u^j_n.$$  \hspace{1cm} (58)

The coefficients $G_{ij}$ and $\hat{H}_{ij}$ are computed again from the integrals (48), but in this case the boundary point $p_i$, is replaced in the expression by the field point $p$ in $\Omega$ (see Figure 6). The line integrals $G_{ij}$ and $\hat{H}_{ij}$ defined in equation (48) are evaluated numerically using a standard Gaussian quadrature \cite{6}. Of course, these integrals can be evaluated numerically through symbolic languages, but the resulting expressions are very lengthy. Hence the advantage of accuracy over the numerical integration is rather lost, due to the complexity of the most suitable method for computing line integrals. Two cases are distinguished for the integrals of the influence coefficients:

(i) Off-diagonal elements, $i \neq j$: In this case, the point $p_i(x_i, y_i)$ lies outside the $j$-element, which means that the distance $r = |r - p_i|$ does not vanish and, consequently, the integral is regular.

The Gaussian integration is performed over the interval $-1 \leq \xi \leq 1$ as

$$\int_{-1}^{1} f(\xi) d\xi = \sum_{k=1}^{n} w_k f(\xi_k)$$  \hspace{1cm} (59)

where $n$ is the number of integration points (Gauss points), and $\xi_k$ and $w_k$ ($k = 1, 2, 3, ..., n$) are the abscissas and weights of the Gaussian quadrature of order $n$.

Let us consider the element $j$ over which the integration will be carried out. This elements is defined by the coordinates $(x_j, y_j)$ and $(x_{j+1}, y_{j+1})$ of its extreme points, which are expressed in a global system having axes $x$ and $y$, and origin at point $O$ (see Figure: 7). Subsequently, a local system of axes $x'$ and $y'$ is introduced at point $p_j$ of the element. The local coordinates $(x', o)$ of point $r$ on the $j$-th element are related to the global coordinates of the $xy$-system through the expressions

$$x = \frac{x_{j+1} + x_j + x_{j+1} - x_j}{2l_j} x', \quad y = \frac{y_{j+1} + y_j + y_{j+1} - y_j}{2l_j} x', \quad -\frac{l_j}{2} \leq x' \leq \frac{l_j}{2}$$  \hspace{1cm} (60)
where $l_j$ is the length of $j$-th element and is given in terms of the coordinates of the end points as

$$l_j = \sqrt{(x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2}. \quad (61)$$

Figure 7: Global and local coordinate systems

Expressions that map the global coordinates onto the integration interval $[-1, +1]$ are obtained by introducing in equation (60) the geometric relation

$$\frac{x'}{\ell_j} = \xi.$$

Thus, the resulting coordinate transformation becomes

$$x(\xi) = \frac{x_{j+1} + x_j}{2} + \frac{x_{j+1} - x_j}{2} \xi, \quad y(\xi) = \frac{y_{j+1} + y_j}{2} + \frac{y_{j+1} - y_j}{2} \xi. \quad (62)$$

Moreover, it is

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{x_{j+1} - x_j}{2}\right)^2 + \left(\frac{y_{j+1} - y_j}{2}\right)^2} \, d\xi = \frac{l_j}{2} \, d\xi. \quad (63)$$

Hence, the Jacobian of the transformation is

$$|J(\xi)| = \frac{l_j}{2}.$$

On the basis of the foregoing, the integrals of the influence coefficients are evaluated numerically in the following way:
(a) The integral of $G_{ij}$

$$G_{ij} = \int_{\Gamma_j} v ds = \int_{-1}^{1} \frac{1}{2\pi} \ln[r(\xi)] \frac{l_j}{2} d\xi = \frac{l_j}{4\pi} \sum_{k=1}^{n} ln[r(\xi_k)]w_k$$  \quad (64)$$

where

$$r(\xi_k) = \sqrt{[x(\xi_k) - x_i]^2 + [y(\xi_k) - y_i]^2}$$  \quad (65)$$

(b) The integral of $\hat{H}_{ij}$: This integral can also be evaluated analytically. From Figure 8,

$$ds \cos \phi = r d\alpha$$

which can be used to derive the expression

$$\hat{H}_{ij} = \int_{\Gamma_j} \frac{1}{2\pi} \cos \phi \frac{r}{r} ds = \int_{\Gamma_j} \frac{1}{2\pi} da = \frac{\alpha_{j+1} - \alpha_j}{2\pi}.$$  \quad (66)$$

The angles $\alpha_{j+1}$ and $\alpha_j$ are computed from

$$\tan \alpha_{j+1} = \frac{y_{j+1} - y_i}{x_{j+1} - x_i}$$  \quad (67)$$

$$\tan \alpha_j = \frac{y_j - y_i}{x_j - x_i}$$  \quad (68)$$

where $x_{j+1}$, $y_{j+1}$ and $x_j$, $y_j$ are the coordinates of the extreme points of the $j$-th element.

Figure 8: Definition of angles involved in the numerical integration over constant elements [4].

(ii) Diagonal elements, $i = j$: In this case, the node $p_i$ coincides with $p_j$, and $r$ lies on the element. Consequently, it is $\phi = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, which yields $\cos \phi = 0$. Moreover, we have

$$x_{p_j} = \frac{x_{j+1} + x_j}{2}, \quad y_{p_j} = \frac{y_{j+1} - y_j}{2}$$
and
\[ r(\xi) = \sqrt{[x(\xi) - x_{p_j}]^2 + [y(\xi) - y_{p_j}]^2} = \frac{l_j}{2}|\xi|. \]  
(69)

Hence,
\[ G_{jj} = \int_{\Gamma_j} \frac{1}{2\pi} \ln r ds = 2 \int_0^{\frac{l_j}{2}} \frac{1}{2\pi} \ln rdr = \frac{1}{\pi} \left[r \ln r - r\right]_0^{\frac{l_j}{2}} = \frac{1}{2} \frac{l_j}{2} \left[\ln \left(\frac{l_j}{2}\right) - 1\right], \] 
(70)

and
\[ \tilde{H}_{jj} = \frac{1}{2\pi} \int_{\Gamma_j} \frac{\cos \phi}{r} ds = \frac{1}{2\pi} \int_{-1}^{1} \frac{\cos \phi}{|\xi|} d\xi = \frac{2}{2\pi} \left[\cos \phi \ln |\xi|\right]_0^{1} = 0. \] 
(71)

4 Results

In this section, we present results to the model in subsection 2.6. We considered different cases of boundary conditions and obtained the hydraulic head at chosen points in the domain.

4.1 Specified head and no flux at Neumann sides

This type of boundary conditions is known as no flow Boundary, it is a special type of the prescribed flux boundary and is also referred to as no-flux, zero flux, impermeable, reflective or barrier boundary. No flow boundaries are impermeable boundaries that allow zero flux. They are physical or hydrological barriers which inhibit the inflow or outflow of water in the model domain, Figure 9. The expressions \( h_1 \) and \( h_2 \) are the values of the hydraulic head at the boundary of the domain. A simulation of the model with these boundary conditions produced the groundwater head distribution shown in Figure 10 and the direction of flow is shown in Figure 11.
4.2 Different flux values at Neumann sides

In Figure 12, we assume the model defined in Section 2.6 with the same values of $h_1$ and $h_2$ at the boundary as in Figure 9 presented in Section 4.1. We assume different values of the flux at Neumann sides as shown in Figure 12. A simulation with these boundary conditions produced the groundwater head distribution shown in Figure 13 and the direction of flow is shown in Figure 14.
Boundary element method of modelling steady state groundwater flow

Figure 12: Undergroundwater flow model of the steady-state with zero flux at one of Neumann sides and non zero flux for another

Figure 13: Surface representation of h levels when $h_1 = 100m$, $h_2 = 50m$, and $\frac{dh}{dn}$ as flow for the model in Figure 12 shown in Figure 12

Figure 14: Gradient representation of the flow for the model in Figure 12.

4.3 Mixed experimental boundary conditions

Now, let us assume some values of the flux at the boundary of the domain bigger than the ones used in Sections 4.1, and 4.2. The values of the flux at Neumann sides are assumed to be 0 and 100 as illustrated in Figure 15. A simulation with these boundary conditions produced the groundwater head distribution shown in Figure 16 and the direction of flow is shown in Figure 17.
Figure 15: Underground water flow model of the steady-state.

Figure 16: Surface representation of level when $h_1 = 100m$, $h_2 = 50m$, and $\frac{dh}{dn}$ as flow for the model in Figure 15 above shown in Figure 15.

Figure 17: Gradient representation of the flow for the model in Figure 15 above.

5 Adjacent Neumann and Dirichlet sides

We are giving different arrangements to the boundary quantities as shown by Figure 19, illustrating the model. The sides with Neumann conditions are adjacent and the Dirichlet sides are also adjacent whereas they are parallel in the precedent sections as illustrated by Figures 9, 12 and 15.

This type of boundary conditions can be used to model a domain with an impermeable wall and a specified hydraulic head boundary as illustrated in Figure 18.
The case illustrated in Figure 18, can be modeled as shown in Figure 19.

A simulation of the model with the boundary conditions shown in Figure 19 produced the groundwater head distributions shown in Figures 20, 22 and 23, and the direction of flow is shown in Figure 21.
At a fixed point 0.5 of \( x \), that is, for all chosen points in the flow domain, which has 0.5 as \( x \) coordinate and \( y \) coordinate varying, the values of the hydraulic head are increasing as the \( y \) values of the chosen points in the domain increase as illustrated by Figure 24. The hydraulic head \( h \) is a function of \( y \) and we write \( h = h(y) \).

When the flow is in one direction as illustrated by Figure 11 in Section 4.1, the values of the hydraulic head are increasing linearly as the \( y \) values increase at chosen point in the flow domain as illustrated by Figure 25 when \( x \) coordinate is fixed at 0.5.

6 Discussion

The following discussion revealed that water flows from points at higher hydraulic head to points with lower hydraulic head which is in agreement with
the general theory of groundwater flow and also with the results obtained in our 2014 paper [8].

The colorbar in Figure 10, show that values of the hydraulic head \( (h) \) in the flow domain are varying from about 55 up to about 95. The smallest value of \( h \) in the flow domain is 54.8m at the points (5/11,1/11) and (6/11,1/11). The highest value of \( h \) in the flow domain is 95.3m at the points (5/11,10/11) and (6/11,10/11). By reducing the distance between the boundary of the domain and the points in the flow domain, the smallest value of \( h \) can be very close to 50m than 54.8m which is obtained by taking the values of \( h \) in the flow domain at 1/11 from the boundary of the domain. The arrows in Figure 11 which represent the gradient vectors show that water is flowing in one direction, from high values of \( h \) to the lowest ones. We can also conclude that the flux is remaining the same as the flux is equal to zero at Neumann sides which is making all the vectors in Figure 11 representing the gradient to have the same magnitude.

We can say that the values of the flux used in Sections 4.1, 4.2 are too small and do not have a significant influence to the results as expected mathematically. Moreover, the values of the hydraulic head and the flux used in this section reflect the reality according to the literature. Hence, we can take our model as experimental which means that the values at the boundary are not necessarily reflecting the reality at the field, by replacing the value 0.0023 of the flux in Figure 12 which is reflecting the reality by 100, which is not used any where in the surveyed literature. The experimental model help us to check the effect of the flux on the result of the model.

The result of this experimental model represented by Figure 15 generated 55.9m as the smallest value of \( h \) and 110.8m. These values of the smallest and biggest values of \( h \) are very different to the ones obtained in Section 4.2 which have the same arrangement of boundary quantities. Hence, we can conclude that the value of the flux at the boundary is has a significant influence on the values of \( h \) in the flow domain. The flow direction illustrated by Figure 17, is also different from the ones illustrated by Figures 11, 14. So, we can conclude that the value of the flux at the boundary of the flow domain influence both the flow direction and the values of \( h \) in the flow domain. Since the flow is coming from out of the model domain which flow domain do not allow the flow at the opposite side where the flux is zero, the quantity of water increases in the flow domain and the value of the hydraulic head also increases up to 110.8m as a maximum value of the head in the flow domain.
Figure 20 illustrates the value of the hydraulic head at any point of the flow domain. The smallest value of $h$ is 99.9985 and the biggest one is 100.0005. The surface representation of values of $h$ at chosen points in the flow domain are distributed on the interval $[99.9985, 100.0005]$ which is a very small range. The color representing values which are on the interval $[99.9995, 100]$ is mainly represented on the Figure 20 which means that at many points in the flow domain, the values of $h$ are on that interval.

The same results represented in Figure 20 which is the surface representation in two dimension, can be represented in three dimension by the Figure 22. We can also represent the values of $h$ in the flow domain in three dimension by Figure 23, when the model is represented by Figure 15.

6.1 Conclusions

In this study, a mathematical model for underground water flow which includes the governing equation and boundary conditions have been formulated and analyzed. The general objective was to develop a prediction tool for underground water flow model using the boundary element method. Two scenarios were considered (when the number of elements at the boundary is fifty six and when it is twenty eight) and two cases in each scenario were considered for generating results. The model was transformed in terms of elements at the boundary of its domain. Since we considered the steady state flow, the fixed boundary values of the head ($h$) at the boundary of the domain took different values to check its influence on values of $h$ at different points in the model domain. The results confirm that water moved from points with high values of $h$ to points with lower values of $h$.

The limitation of the model considered here is that we assumed regular boundaries with artificial boundary conditions, which may slightly deviate from reality.

The concluded research leaves rich areas for further research in the direction of solute transport models and a consideration of irregular boundaries with experimental data.
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