Domination Number of the Acquaint Vertex Gluing of Graphs

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Abstract

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a dominating set of $G$ if for every $x \in V \setminus S$, there exists $y \in S$ such that $xy \in E$.

In this paper, we introduced a new binary graph operation which we call acquaint vertex gluing. Moreover, we gave the domination number of the acquaint vertex gluing of some graphs.

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1 Introduction

This paper would like to determine the domination number some graphs resulting from the acquaint vertex gluing of paths, cycles, complete graphs and wheels.

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The path $P_n = (v_1, v_2, \ldots, v_n)$ is the graph with distinct vertices $v_1, v_2, \ldots, v_n$ and edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$. The cycle $C_n = [v_1, v_2, \ldots, v_n]$, $n \geq 3$, is the graph with vertices $v_1, v_2, \ldots, v_n$ and edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$. A complete graph of order $n$, denoted by $K_n$, is the graph in which every pair of distinct vertices are adjacent.

Let $X$ and $Y$ be sets. The disjoint union of $X$ and $Y$, denoted by $X \cup Y$, is found by combining the elements of $X$ and $Y$, treating all elements to be distinct. Thus, $|X \cup Y| = |X| + |Y|$. The join of two graphs $G$ and $H$, denoted by $G + H$, is the graph with vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The wheel $W_n$ is the join of $K_1 = (\{v\}, \emptyset)$ and $C_n$, that is, $W_n = K_1 + C_n$.

Let $G = (V, E)$ be a graph and $S \subseteq V$. The open neighborhood of $S$, denoted by $N(S)$, is consists of all vertices $y \in V$ which are adjacent to some element $x \in S$. Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a dominating set of $G$ if for every $x \in V \setminus S$, there exists $y \in S$ such that $xy \in E$, that is, $N_G[S] = S \cup N_G(S) = V$. The domination number of $G$, denoted by $\gamma(G)$, is the cardinality of a minimum dominating set.

![Figure 1: A graph $G$](image)

Consider the graph $G$ in Figure 1 and let $S = \{a, d\}$. Then $N(a) = \{b, d\}$ and $N(d) = \{a, b, c, e\}$. Hence, $N(S) = N(a) \cup N(d) = \{b, d\} \cup \{a, b, c, e\} = \{a, b, c, d, e\}$. Thus, $N[S] = S \cup N(S) = \{a, d\} \cup \{a, b, c, d, e\} = \{a, b, c, d, e\} = V(G)$. Therefore, $S$ is a dominating set of $G$. Similarly, $S' = \{d\}$ is a dominating set. Clearly, $S'$ is a minimum dominating set. So, $\gamma(G) = 1$.

**Definition 1.1** Let $m, n \in \mathbb{N}$ and $P_n = (1, 2, \ldots, n)$ be a path of order $n$. Let

$$P_n^{(1)} = \left(1^{(1)}, 2^{(1)}, \ldots, n^{(1)}\right)$$

$$P_n^{(2)} = \left(1^{(2)}, 2^{(2)}, \ldots, n^{(2)}\right)$$

$$\vdots$$

$$P_n^{(m)} = \left(1^{(m)}, 2^{(m)}, \ldots, n^{(m)}\right).$$

be the $m$ copies of $P_n$ with $m \geq 3$. The acquaint vertex gluing of the $m$ copies of $P_n$, denoted by $mP_n$, is the graph obtained by identifying the vertices, $1^{(1)}$,
1\(^{(2)}\), ..., 1\(^{(m)}\), and connecting by an edge the vertices,

\[ n^{(1)} \text{ and } n^{(2)} \]
\[ n^{(2)} \text{ and } n^{(3)} \]
\[ \vdots \]
\[ n^{(m-1)} \text{ and } n^{(m)} \]
\[ n^{(m)} \text{ and } n^{(1)} \].

**Definition 1.2** Let \( m, n \in \mathbb{N} \) and \( C_n = [1, 2, \ldots, n] \) be a cycle of order \( n \). Let

\[ C_n^{(1)} = [1^{(1)}, 2^{(1)}, \ldots, n^{(1)}] \]
\[ C_n^{(2)} = [1^{(2)}, 2^{(2)}, \ldots, n^{(2)}] \]
\[ \vdots \]
\[ C_n^{(m)} = [1^{(m)}, 2^{(m)}, \ldots, n^{(m)}] \]

be the \( m \) copies of \( C_n \) with \( m \geq 3 \). The *acquaint vertex gluing* of the \( m \) copies of \( C_n \), denoted by \( mC_n \), is the graph obtained by identifying the vertices, \( 1^{(1)}, 1^{(2)}, \ldots, 1^{(m)} \) and connecting by an edge the vertices,

\[ \left( \frac{n + 2}{2} \right)^{(1)} \text{ and } \left( \frac{n + 2}{2} \right)^{(2)} \]
\[ \left( \frac{n + 2}{2} \right)^{(2)} \text{ and } \left( \frac{n + 2}{2} \right)^{(3)} \]
\[ \vdots \]
\[ \left( \frac{n + 2}{2} \right)^{(m-1)} \text{ and } \left( \frac{n + 2}{2} \right)^{(m)} \]
\[ \left( \frac{n + 2}{2} \right)^{(m)} \text{ and } \left( \frac{n + 2}{2} \right)^{(1)} \].

if \( n \) is even, while we connect by an edge the vertices,

\[ \left\lfloor \frac{n}{2} \right\rfloor^{(1)} \text{ and } \left\lfloor \frac{n + 2}{2} \right\rfloor^{(2)} \]
\[ \left\lfloor \frac{n}{2} \right\rfloor^{(2)} \text{ and } \left\lfloor \frac{n + 2}{2} \right\rfloor^{(3)} \]
\[ \vdots \]
\[ \left\lfloor \frac{n}{2} \right\rfloor^{(m-1)} \text{ and } \left\lfloor \frac{n + 2}{2} \right\rfloor^{(m)} \]
\[ \left\lfloor \frac{n}{2} \right\rfloor^{(m)} \text{ and } \left\lfloor \frac{n + 2}{2} \right\rfloor^{(1)} \].

if \( n \) is odd.
Definition 1.3 Let \( m, n \in \mathbb{N} \) with \( m \geq 3 \), and let \( K_n^{(1)}, K_n^{(2)}, \ldots, K_n^{(m)} \). Let \( u^{(i)}, v^{(i)} \in K_n^{(i)} \) with \( u^{(i)} \neq v^{(i)}, i = 1, 2, 3, \ldots, m \). The **acquaint vertex gluing** of \( m \) copies of \( K_n \), denoted by \( mK_n \), is the graph obtained by identifying the vertices \( u_i, i = 1, 2, 3, \ldots, m \) and connecting by an edge the pair of vertices

\[
\begin{align*}
v^{(1)} \text{ and } v^{(2)} \\
v^{(2)} \text{ and } v^{(3)} \\
\vdots \\
v^{(m-1)} \text{ and } v^{(m)} \\
v^{(m)} \text{ and } v^{(1)}.
\end{align*}
\]

Definition 1.4 Let \( m, n \in \mathbb{N} \) with \( m \geq 3 \) and \( W_n = K_1 + [1, 2, \ldots, n] \) be a wheel of order \( n + 1 \). Let

\[
\begin{align*}
W_n^{(1)} &= K_1^{(1)} + [1^{(1)}, 2^{(1)}, \ldots, n^{(1)}] \\
W_n^{(2)} &= K_1^{(2)} + [1^{(2)}, 2^{(2)}, \ldots, n^{(2)}] \\
&\vdots \\
W_n^{(m)} &= K_1^{(m)} + [1^{(m)}, 2^{(m)}, \ldots, n^{(m)}].
\end{align*}
\]

be the \( m \) copies of \( W_n \). The **acquaint vertex gluing** of the \( m \) copies of \( W_n \), denoted by \( mW_n \), is the graph obtained by identifying the vertices, \( 1^{(1)}, 1^{(2)}, \ldots, 1^{(m)} \) and connecting by an edge the pair of vertices

\[
\begin{align*}
\left( \frac{n + 2}{2} \right)^{(1)} \text{ and } \left( \frac{n + 2}{2} \right)^{(2)} \\
\left( \frac{n + 2}{2} \right)^{(2)} \text{ and } \left( \frac{n + 2}{2} \right)^{(3)} \\
&\vdots \\
\left( \frac{n + 2}{2} \right)^{(m-1)} \text{ and } \left( \frac{n + 2}{2} \right)^{(m)} \\
\left( \frac{n + 2}{2} \right)^{(m)} \text{ and } \left( \frac{n + 2}{2} \right)^{(1)}.
\end{align*}
\]
if $n$ is even, while we connect by an edge the vertices,
\[
\begin{align*}
\left\lceil \frac{n}{2} \right\rceil & \text{ and } \left\lceil \frac{n+2}{2} \right\rceil^{(2)} \\
\left\lceil \frac{n}{2} \right\rceil & \text{ and } \left\lceil \frac{n+2}{2} \right\rceil^{(3)} \\
& \quad \vdots \phantom{\text{ and } \left\lceil \frac{n+2}{2} \right\rceil^{(3)}} \\
\left\lceil \frac{n}{2} \right\rceil & \text{ and } \left\lceil \frac{n+2}{2} \right\rceil^{(m)} \\
\left\lfloor \frac{n}{2} \right\rfloor & \text{ and } \left\lfloor \frac{n+2}{2} \right\rfloor^{(1)}.
\end{align*}
\]
if $n$ is odd.

As found in [1], the Pigeon-hole Principle states that if $k$ and $n$ be any two positive integers and if at least $kn+1$ objects are distributed among $n$ boxes, then one of the boxes must contain at least $k+1$ objects. In particular, if at most $n-1$ objects are to be put into $n$ boxes, then one of the boxes is empty.

For the notations and concepts in the succeeding sections which are not discussed above, please refer to [2] and [3].

The following sections present the main results of this study.

## 2 Domination Number of the Acquaint Vertex Gluing of Path

**Theorem 2.1** Let $r, k \in \mathbb{N}$. Then $\delta_{(3^r P_{3k})} = 3rk - 2r + 1$.

**Proof:** Let $P_{3k} = (1, 2, \ldots, 3k)$ and the $P_{3k}^{(1)} = (1^{(1)}, 2^{(1)}, \ldots, 3k^{(1)}),$ $P_{3k}^{(2)} = (1^{(2)}, 2^{(2)}, \ldots, 3k^{(2)}), \ldots,$ $P_{3k}^{(3r)} = (1^{(3r)}, 2^{(3r)}, \ldots, 3k^{(3r)}),$ be the $m$ copies of $P_{3k}$. Form $3^r P_{3k}$ by identifying vertices $1^{(1)}, 1^{(2)}, \ldots, 1^{(3r)}$. Consider $S = \{1^{(1)}, 4^{(1)}, 4^{(2)}, \ldots, 4^{(3r)}, 7^{(1)}, 7^{(2)}, \ldots, 7^{(3r)}, \ldots, (3k-2)^{(1)}, (3k-2)^{(2)}, \ldots, (3k-2)^{(3r)}, (3k)^{(1)}, (3k)^{(2)}, (3k)^{(3)}, \ldots, (3k)^{(3r-2)}\}$.

Note that $N \left[ 1^{(1)} \right] = \{1^{(1)}, 2^{(1)}, 2^{(2)}, 2^{(3)}, \ldots, 2^{(3r)} \}$, $N \left[ 4^{(1)} \right] = \{3^{(1)}, 4^{(1)}, 5^{(1)} \}$, $N \left[ 4^{(2)} \right] = \{3^{(2)}, 4^{(2)}, 5^{(2)} \}, \ldots$, $N \left[ 4^{(3r)} \right] = \{3^{(3r)}, 4^{(3r)}, 5^{(3r)} \}$, $N \left[ 7^{(1)} \right] = \{6^{(1)}, 7^{(1)}, 8^{(1)} \}$, $N \left[ 7^{(2)} \right] = \{6^{(2)}, 7^{(2)}, 8^{(2)} \}, \ldots$, $N \left[ 7^{(3r)} \right] = \{6^{(3r)}, 7^{(3r)}, 8^{(3r)} \}$, $\ldots$, $N \left[ (3k-2)^{(1)} \right] = \{(3k-2)^{(1)}, (3k-2)^{(1)}, (3k-1)^{(1)} \}$, $N \left[ (3k-2)^{(2)} \right] = \{(3k-3)^{(2)}, (3k-2)^{(2)}, (3k-1)^{(2)} \}, \ldots$, $N \left[ (3k-2)^{(3r)} \right] = \{(3k-3)^{(3r)}, (3k-2)^{(3r)}, (3k-1)^{(3r)} \}$, $N \left[ (3k)^{(1)} \right] = \{(3k)^{(3r)}, (3k)^{(1)}, (3k)^{(2)} \}$, $N \left[ (3k)^{(4)} \right] = \ldots$
Let $m, n \in \mathbb{N}$. If $n \neq 3k$ for any $k \in \mathbb{N}$, then $\delta(mP_n) = m \left\lceil \frac{n-2}{3} \right\rceil + 1$.

**Proof:** Let $P_n = (1, 2, \ldots, n)$ and the $P_n^{(1)} = (1^{(1)}, 2^{(1)}, \ldots, n^{(1)})$, $P_n^{(2)} = (1^{(2)}, 2^{(2)}, \ldots, n^{(2)})$, ..., $P_n^{(m)} = (1^{(m)}, 2^{(m)}, \ldots, n^{(m)})$ be the $m$-copies of $P_n$.

Form $mP_n$ by identifying vertices $1^{(1)}, 1^{(2)}, \ldots, 1^{(m)}$.

If $m \equiv 2 \pmod{3}$, then $S = \{1^{(1)}, 4^{(1)}, 4^{(2)}, \ldots, 4^{(m)}, 7^{(1)}, 7^{(2)}, \ldots, 7^{(m)}, \ldots, (n-1)^{(1)}, (n-1)^{(2)}, \ldots, (n-1)^{(m)} \}$ is a dominating set.

To see this, we note that $N(1^{(1)}) = \{2^{(1)}, 2^{(2)}, 2^{(3)}, \ldots, 2^{(m)} \}$, $N(4^{(1)}) = \{3^{(1)}, 5^{(1)} \}$, $N(4^{(2)}) = \{3^{(2)}, 5^{(2)} \}$, ..., $N(4^{(m)}) = \{3^{(m)}, 5^{(m)} \}$, $N(7^{(1)}) = \{6^{(1)}, 8^{(1)} \}$, $N(7^{(2)}) = \{6^{(2)}, 8^{(2)} \}$, ..., $N(7^{(m)}) = \{6^{(m)}, 8^{(m)} \}$, ..., $N((n-1)^{(1)}) = \{(n-2)^{(1)}, n^{(1)} \}$, $N((n-1)^{(2)}) = \{(n-2)^{(2)}, n^{(2)} \}$, ..., $N((n-1)^{(m)}) = \{(n-2)^{(m)}, n^{(m)} \}$.

Thus, $N[S] = N(S) \cup S = \{2^{(1)}, 2^{(2)}, 2^{(3)}, \ldots, 2^{(m)}, 3^{(1)}, 5^{(1)}, 3^{(2)}, 5^{(2)}, \ldots, 3^{(m)}, 5^{(m)}\}, \{6^{(1)}, 8^{(1)}\}, \{6^{(2)}, 8^{(2)}\}, \ldots, \{6^{(m)}, 8^{(m)}\}, \{(n-2)^{(1)}, n^{(1)}\}, \{(n-2)^{(2)}, n^{(2)}\}, \ldots, \{(n-2)^{(m)}, (n-1)^{(m)}\}$ of $V(mP_n)$. If $\delta(mP_n) < m \left\lceil \frac{n-2}{3} \right\rceil + 1$ and $S'$
is a dominating set with $|S'| = \delta(mP_n)$, then by the Pigeon-hole Principle at least one element of the partition will not contain an element of $S'$. This is a contradiction since $S'$ is a dominating set. Therefore, $\delta(mP_n) = m \left\lceil \frac{n-2}{3} \right\rceil + 1$.

Similarly, if $m \equiv 1 \pmod{3}$, then $S = \{1^{(1)}, 4^{(1)}, 4^{(2)}, \ldots, 4^{(m)}, 7^{(1)}, 7^{(2)}, \ldots, 7^{(m)}, \ldots, (n-1)^{(1)}, (n-1)^{(2)}, \ldots, (n-1)^{(m)}\}$ is a minimum dominating set. Note that $|S| = m \left\lceil \frac{n-2}{3} \right\rceil + 1$. Therefore, $\delta(mP_n) = m \left\lceil \frac{n-2}{3} \right\rceil + 1$. □

**Theorem 2.3** Let $m, k \in \mathbb{N}$. If $m \neq 3k$ for any $k \in \mathbb{N}$ and $3 \mid \left\lceil \frac{n}{2} \right\rceil$, then $\delta(mP_{3k}) = \left\lceil \frac{m}{3} \right\rceil + m(k - 1) + 1$.

### 3 Domination Number of the Acquaint Vertex

**Gluing of Complete Graphs**

**Theorem 3.1** Let $m, n \in \mathbb{N}$ with $m \geq 3$ and $n \geq 2$. Then $\delta(mK_n) = 1$.

**Proof:** Let $K_n$ be a complete graph of order $n$. Let $u \in V(mK_n)$ with $\deg_{mK_n}(u) = mn$, then $S = \{u\}$ is a dominating set in $mK_n$. Hence, $\delta(mK_n) \leq 1$. Since $\delta(mK_n) \geq 1$, we must have $\delta(mK_n) = 1$. □

### 4 Domination Number of the Acquaint Vertex

**Gluing of Cycles**

**Corollary 4.1** Let $m, n \in \mathbb{N}$ with $m \geq 3$. If $n$ is odd, then

$$\delta(mC_n) = \begin{cases} 
2m \left\lceil \frac{n}{3} \right\rceil - \frac{4m}{3} + 1, & \text{if } 3 \mid 2m \text{ and } 3 \mid \left\lceil \frac{n}{2} \right\rceil \\
2m \left\lceil \frac{n}{3} - 2 \right\rceil + 1, & \text{if } 3 \nmid \left\lceil \frac{n}{2} \right\rceil \\
\left\lceil \frac{2m}{3} \right\rceil + \frac{2m}{3} - 2m + 1, & \text{if } 3 \nmid 2m \text{ and } 3 \mid \left\lceil \frac{n}{2} \right\rceil.
\end{cases}$$

**Proof:** Since $mC_n \cong 2mP_{\left\lceil \frac{n}{2} \right\rceil}$, the result follows from Theorems 2.1, 2.2 and 2.3. □

**Theorem 4.2** Let $m, k \in \mathbb{Z}$. Then

1. $\delta(mC_{6k}) = 2mk - m + 1$;
2. If $k \equiv 2 \pmod{3}$, then $\delta(3mC_{2k}) = 2mk - 3m + 1$. 

5 Domination Number of the Acquaint Vertex Gluing of Wheels

Theorem 5.1 Let $m, n \in \mathbb{N}$ with $m \geq 3$, then $\delta(mW_n) = m$.

Proof: Let $W_n = (\{u\}, \emptyset) + [1, 2, \ldots, n]$ a wheel of order $n+1$. Let the following be $m$ copies of $W_n$.

\[ W_n^{(1)} = (\{u^{(1)}\}, \emptyset) + [1^{(1)}, 2^{(1)}, \ldots, n^{(1)}] \]
\[ W_n^{(2)} = (\{u^{(2)}\}, \emptyset) + [1^{(2)}, 2^{(2)}, \ldots, n^{(2)}] \]
\[ \vdots \]
\[ W_n^{(m)} = (\{u^{(m)}\}, \emptyset) + [1^{(m)}, 2^{(m)}, \ldots, n^{(m)}] \]

Form $mW_n$ by identifying vertices $1^{(1)}, 1^{(2)}, \ldots, 1^{(m)}$. Let $S = \{u^{(1)}, u^{(2)}, \ldots, u^{(m)}\}$. Then clearly $S$ is a dominating set in $mW_n$. Hence, $\delta(mW_n) \leq m$.

Suppose $\delta(mW_n) < m$. Let $S = \{v_1, v_2, v_3, \ldots, v_k\}$ with $k < m$ be a dominating set in $mW_n$. Let $v = 1^{(1)}$ and consider the following cases.

Case 1. $v \in S$

If $v \in S$, then $v = v_i$ for some $i \in \{1, 2, \ldots, k\}$. Thus, there will be only $k - 1$ guards for the other vertices. Since $k < m$, it follows that $k - 1 < m$. Hence by Pigeonhole Principle, there exists a copy of $W_n$, say $W_n^{(j)}$, such that $S \cap (V(W_n^{(j)}) \setminus \{v\}) = \emptyset$. Thus, $3^{(j)} \notin N[S]$. This is a contradiction since $S$ is a dominating set.

Case 2. $v \notin S$

If $v \notin S$ and $k < m$, then by Pigeonhole Principle, there exists a copy of $W_n$, say $W_n^{(j)}$, such that $S \cap V(W_n^{(j)}) = \emptyset$. Hence, $3^{(j)} \notin N[S]$. This is a contradiction since $S$ is a dominating set.

Accordingly, $\delta(mW_n) = m$. \hfill \Box

References


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