Mathematical Model of Pulsating Viscous Liquid Layer Movement in a Flat Channel with Elastically Fixed Wall

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Abstract

The problem of dynamic interaction of elastically fixed channel wall with pulsating viscous incompressible liquid layer is set up and analytically solved. The problem in a flat setting for the regime of a stationary pulsating liquid movement in the cannel under the suggested harmonic law of pressure pulsating at its butt end is considered. The formulated bound problem represents non-linear connected Navier-Stokes equations system for viscous incompressible liquid layer and the equation of elastically fixed channel wall dynamics. The conditions of liquid adhesion to impenetrable channel walls and the condition of free leakage of liquid at channel butt ends are presented in the paper as the bound ones. The complex of dimensionless variables of the problem under consideration and its small parameters are singled out well. The relative thickness of liquid layer and relative amplitude of channel wall fluctuations are taken as small parameters. The linearization of the problem by means of perturbation method is made in the course...
of asymptotic expansions, according to the small parameters, singed out above. The solution of the linearized problem is made by means of the assigned forms method for adjusted harmonic fluctuations. The law of elastically fixed channel wall shift and hydrodynamic parameters distribution in liquid are defined. Frequency dependent function of dynamic pressure distribution along the channel, frequency dependent function of phase shift pressure in the channel, amplitude frequency characteristics and phase frequency characteristics of elastically fixed channel wall in relation to the initial perturbation in the butt end are received.

**Keywords:** mathematical modeling, hydroelasticity, viscous liquid, flat channel, pressure pulsating, fluctuations, elastically fixed wall

**Introduction**

The research in liquid movement in flat channel problem is quite frequent in consideration of numerous hydrodynamics and hydroelasticity problems [5, 6, 10, 15, 14]. The dynamics of viscous incompressible liquid layer in the channel with absolutely solid walls is investigated in the frames of hydrodynamic lubrication theory [10, 15]. Paper [9] presents the experimental study of periodically pulsating viscous liquid movement stability in flat channel with immovable solid walls.

Nevertheless, the investigation of the causes and conditions of vibration cavitations emergence in a liquid inside the channel remain actual, as well. For example, [7, 8] present vibration cavitations in cooling liquid of internal-combustion engines. Paper [8] gives the experimental basis of the fact, that channel walls vibration is the main cause of vibration cavitations. The channel wall fluctuations are considered from the point of view of their being free fluctuations of cylinder shell, without taking into account its interaction with liquid and its viscosity influence. Paper [7] researches cavitations emergence on the basis of elastic beam with ideal liquid interaction problem solution. However, liquid viscosity turns out to be of importance as its (liquid viscosity) determines damping characteristics in the fluctuation system under consideration, and, therefore, the amplitude fluctuations of channel walls limit. Paper [2] presents the investigation of hydroelastic beam fluctuations in the viscous liquid flow with application piezoelectric element, which can be used for receiving energy from the flow.

On the other hands, the problem of vibration discs interaction with viscous incompressible liquid layer between them, including case, when one disc presents a circular elastic plate was solved in [12]. It is shown, that under resonance frequencies of channel walls fluctuations, liquid pressure can become substantially smaller than saturated vapor pressure. The analogical research in flat setting for plates is made in [11], and for circular three layer plates in [1]. The vibration of circular plate on free surface of ideal incompressible liquid is considered
in [3]. It should be noted, that only the liquid part limited by a solid bottom and a cylinder wall is studied in this paper.

The circular plate fluctuations, plunged into the water, with a free surface are investigated in [4].

The paper considers the possibilities of vibration cavitations emergence in viscous liquid. To achieve this goal, the problem of movement of pulsating viscous liquid layer, interacting with elastically fixed wall of flat channel, was solved. The peculiarities of butt ends leakage were into account, as well.

**Mathematical Formulation**

Let us consider the channel in the figure. It is formed by two parallel impenetrable solid walls 1, 2 of identical geometric sizes with a pulsating layer of viscous incompressible liquid 3, moving at the expense of the previously suggested law of pressure change at the butt ends. The wall 1 is connected elastically with the channel founding and can move in a vertical direction. The geometrical size of the channel 2ℓ is considerable smaller than it’s size b, and the liquid layer thickness (the distance between the walls) represents an undisturbed state δ₀ and is considerably smaller than 2ℓ. As a result of pressure pulsating, there emerge wall 1 fluctuations, its amplitude shift being considerably smaller than δ₀. The leakage at butt ends can be considered as a spurt one in the cavity, filled with the same liquid. To be definite, let us consider, that the pressure in a left cavity is constant p₀, and in a right cavity it includes a constant component and, also, a component, harmonically changing in time p₀ + p⁺(ωt).

**Figure. The scheme of the flat channel with an elastically fixed wall**

The law of pressure change at the right butt end is presented as:

\[ p^+ = p_m^+ f_p(\omega t), \quad f_p(\omega t) = \exp(i \omega t), \]  \(1\)

where \(p_m^+\) – amplitude of pressure pulsating at butt ends; \(\omega\) – pulsating frequency; \(f_p(\omega t)\) – pressure change law.
Later, we take into account, that in a given mechanical system strong damping determined by liquid viscosity record, takes place. In its turn, in the course of time damping causes a sufficiently quick going out of transient processes. In this case, the influence of initial conditions ceases to influence and harmonic oscillations emerge. Therefore, from the very beginning of the research in lengthy processes of general solution of heterogeneous equations, initial conditions will be excluded [13].

Let us introduce Cartesian coordinates system \( x, y, z \), connected with an absolutely solid wall 2. Taking into account, that \( 2\ell << b \), we consider the channel to be unlimited in axis direction \( y \), that is we start solving a flat problem. In this case the dynamic equation of viscous incompressible liquid in channel are as follows

\[
\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_z \frac{\partial V_x}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial z^2} \right),
\]

\[
\frac{\partial V_z}{\partial t} + V_x \frac{\partial V_z}{\partial x} + V_z \frac{\partial V_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial z^2} \right) - \frac{\partial V_x}{\partial x} + \frac{\partial V_z}{\partial z} = 0,
\]

where \( p \) – pressure, \( \rho \), \( \nu \) – density and kinematical coefficient of liquid viscosity, \( V_x, V_z \) – velocity movement projections on coordinates axis.

Liquid dynamic equations are expanded by bound conditions: liquid adhesion to channel walls [6, 12, 11, 15]

\[
V_x = 0, \ V_z = 0 \text{ at } z = 0, \quad V_x = 0, \ V_z = \frac{dz_1}{dt} \text{ at } z = \delta_0 + z_{im} f(\omega t),
\]

and the conditions of its free butt ends leakage in the from of coincidence of liquid pressure in the cavity

\[
p = p_0 + p^*(\omega t) \text{ at } x = \ell, \quad p = p_0 \text{ at } x = -\ell.
\]

Here \( z_1 = z_0 + z_{im} f(\omega t) \) – the law of channel wall shift; \( z_0 \) – static wall shift at the expense of pressure \( p_0 \); \( z_{im} \) – oscillation amplitude of channel wall.

The equation of the channel wall movement is as follows

\[
m_1 \frac{d^2z_1}{dt^2} + n_1 z_1 = N,
\]

where \( m_1 \) – wall mass, \( n_1 \) – elastic wall suspension rigidity, \( N \) – force, acting from liquid pulsating layer. The expression for force \( N \) takes the form of
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\[ N = \int_a^b \int_{-l}^l q_{zz} \, dx \, dy, \text{ at } z = \delta_0 + z_{im} f(\omega t), \]  \hspace{1cm} (6)

where \( q_{zz} = -p + 2\rho \nu (\partial V_z / \partial z) \) – normal tension, acting from liquid pulsating layer on channel wall.

Let us introduce in our research dimensionless variable

\[ \psi = \delta_0 / \ell \ll 1, \quad \lambda = z_{im} / \delta_0 \ll 1, \quad \tau = \omega t, \quad \xi = x / \ell, \quad \zeta = z / \delta_0, \quad V_z = z_{im} \omega U_z, \]  \hspace{1cm} (7)

Here \( \psi, \lambda, \text{Re} \) – the parameters, describing the problem.

Placing the introduced parameters (7) into the problem (1)-(6), we get the flat channel hydroelasticity problem in dimensionless form, including viscous incompressible liquid equations

\[ \begin{align*}
\text{Re} \left[ \frac{\partial U_\xi}{\partial \tau} + \lambda \left( U_\xi \frac{\partial U_\xi}{\partial \xi} + U_\zeta \frac{\partial U_\zeta}{\partial \zeta} \right) \right] & = -\frac{\partial P}{\partial \xi} + \psi^2 \frac{\partial^2 U_\xi}{\partial \xi^2} + \frac{\partial^2 U_\xi}{\partial \zeta^2}, \\
\psi^2 \text{Re} \left[ \frac{\partial U_\zeta}{\partial \tau} + \lambda \left( U_\xi \frac{\partial U_\zeta}{\partial \xi} + U_\zeta \frac{\partial U_\zeta}{\partial \zeta} \right) \right] & = -\frac{\partial P}{\partial \zeta} + \psi^2 \left[ \frac{\partial^2 U_\zeta}{\partial \xi^2} + \frac{\partial^2 U_\zeta}{\partial \zeta^2} \right], \\
\frac{\partial U_\xi}{\partial \xi} + \frac{\partial U_\zeta}{\partial \zeta} & = 0,
\end{align*} \]  \hspace{1cm} (8)

and the equation of channel wall dynamics

\[ m_1 \frac{d^2 z_i}{dt^2} + n_i \frac{z_i}{2} = 2b \ell p_0 \frac{b \ell \rho \nu z_{im} \omega^2}{\delta_0 \nu^2} \left[ P - 2\psi^2 \frac{\partial U_\zeta}{\partial \zeta} \right] dz_i. \]  \hspace{1cm} (9)

The bound conditions (3) and (4) take the form of

\[ \begin{align*}
U_\xi(0, \zeta) & = 0, \quad U_{\zeta}(0, \zeta) = df(\tau)/d\tau \text{ at } \zeta = 1 + \lambda f(\tau), \\
U_\xi(0, 0) & = 0, \quad U_{\zeta}(0, 0) = 0 \text{ at } \zeta = 0, \\
P(0) & = P^+ = 0 \quad \text{at } \zeta = -1.
\end{align*} \]  \hspace{1cm} (10, 11)

**Solutions and Discussions**

In the suggested problem setting \( \psi \) is a dimensionless small parameter and therefore in a zero approximation with \( \psi \) the equations (8), (9) are simplified. It means, that the members of the order \( \psi \) and \( \psi^2 \) may be considered equaling
zero. Then while considering asymptotic expansions by a small parameter $\lambda \ll 1$ \cite{16} $P = P_0 + \lambda P_1 + ...$, $U_\zeta = U_\zeta^0 + \lambda U_\zeta^1 + ... U_\zeta = U_\zeta^0 + \lambda U_\zeta^1 + ...$ and limiting by only one member of the expansion, we get linearized problem of a hydroelasticity, including the equations of liquid layer dynamics

$$\text{Re} \frac{\partial U_\zeta}{\partial \tau} = - \frac{\partial P_0}{\partial \xi} + \frac{\partial^2 U_\zeta}{\partial \xi^2}, \quad \frac{\partial P_0}{\partial \zeta} = 0, \quad \frac{\partial U_\zeta}{\partial \xi} + \frac{\partial U_\zeta}{\partial \zeta} = 0 \quad (12)$$

with bound conditions

$$U_\zeta = 0, \quad U_\zeta^0 = df / d\tau \text{ at } \zeta = 1, \quad U_\zeta^0 = 0, \quad U_\zeta^0 = 0 \text{ at } \zeta = 0, \quad (13)$$

$$P_0 = P^* (\tau) \text{ at } \zeta = 1, \quad P_0 = 0 \text{ at } \zeta = -1, \quad (14)$$

and the equation of wall

$$m_1 \frac{d^2 z_1}{dt^2} + n_1 z_1 = 2b\ell p_0 + b\ell \rho z_{in} \omega (\delta_0 \psi^2)^{-1} \int_{-1}^{1} P_0 d\zeta. \quad (15)$$

The solution of the equations system (12) with bound conditions (13), (14) in the suggestion of the harmonic character time change of hydrodynamic parameters and channel wall shift are found in the form of

$$U_\zeta = \frac{1}{2\varepsilon^2} \left[ \frac{\partial^2 P_0}{\partial \zeta^2} + \frac{\partial^2 P_0}{\partial \xi^2} \Psi(\zeta) + \frac{\partial P_0}{\partial \zeta} \Phi(\zeta) \right], \quad (16)$$

$$U_\zeta^0 = \frac{1}{2\varepsilon^2} \left[ \frac{\partial^3 P_0}{\partial \xi^2 \partial \tau} (1 - \zeta) + \frac{\partial^3 P_0}{\partial \xi^2 \partial \tau} \Psi(\zeta) + \frac{\partial^2 P_0}{\partial \xi^2} \Phi(\zeta) \right] + \frac{df}{d\tau},$$

$$P_0 = [(\zeta^2 - 1)(2\varepsilon^2 \alpha^2 d^2 f / d\tau^2 + 12\gamma df / d\tau + P^*(\zeta + 1))/2,$$

where $\varepsilon(\omega) = \sqrt{\text{Re}/2}$, $\Phi(\zeta) = F_2 (\alpha \zeta) D_1 - F_1 (\alpha \zeta) - 2F_3 (\alpha \zeta) D_2,$

$$\Phi(\zeta) = 2F_3 (\alpha \zeta) - F_3 (\alpha \zeta) D_2 - 2F_4 (\alpha \zeta) D_1, \quad \Phi(\zeta) = \int_{-1}^{1} \Phi(\zeta) \text{d}\zeta,$$

$$F_3 (\alpha \zeta) = \text{ch} \alpha \zeta \cos \alpha \zeta, \quad F_3 (\alpha \zeta) = \text{ch} \alpha \zeta \sin \alpha \zeta / 2; \quad F_4 (\alpha \zeta) = \text{sh} \alpha \zeta \sin \alpha \zeta / 2,$$

$$D_1 = (\sin \varepsilon - \sin \varepsilon) / (\cos \varepsilon + \cos \varepsilon), \quad D_2 = (\sin \varepsilon + \sin \varepsilon) / (\cos \varepsilon + \cos \varepsilon),$$

$$\gamma(\omega) = \frac{1}{6 \varepsilon^3 (\text{ch} \varepsilon - \sin \varepsilon)}.$$
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\[ \alpha(\omega) = \frac{e(\cos \varepsilon + \cos \varepsilon) - (\cos \varepsilon + \sin \varepsilon)}{e^2(\cos \varepsilon + \cos \varepsilon) - 2e(\cos \varepsilon + \sin \varepsilon) + 2(\cos \varepsilon - \cos \varepsilon)}. \]

The equation (15) in consideration of (16) take the form of

\[ (m_i + M) \frac{d^2 z_i}{dt^2} + 2K \frac{d z_i}{dt} + n_i z_i = 2b \frac{dp_0}{dt} + C^+ p^+, \]  
(17)

where \( 2K = 8 \ell \beta \nu (\delta \gamma / \nu^2)^{-1} \gamma, \ M = 4 \ell \beta \nu (\omega \delta \gamma / \nu^2)^{-1} \epsilon^2 \alpha / 3. \)

The solution of the equation (17) under the law of pressure change in time (1) takes the form of

\[ z_i = z_0 + z_{im} f(\tau) = 2 \beta p_0 / n_i + p_m^+ (b A(\omega) \exp [i(\omega t + \varphi_0(\omega))]), \]  
(18)

where \( z_0 = 2 \beta p_0 / n_i, \ f(\tau) = p_m^+ b (A_p f_p + B_p d_f_p / d\tau) / z_{im} \), \( A_p = n_i - (m_i + M) \omega^2 \) \[ n_i - (m_i + M) \omega^2 \] \[ + \left( 2K \omega \right), \ B_p = - \frac{2K \omega}{\left( 2K \omega \right)}, \ A(\omega) = \sqrt[4]{\left( n_i - (m_i + M) \omega^2 \right)} + \left( 2K \omega \right), \tg \varphi_0(\omega) = 2K \omega / ((m_i + M) \omega^2 - n_i).

Using the newly found expression for the wall shift (18), we can finally define the law of pressure change dynamics along the channel in a dimensional form

\[ p = p^+(\tau)(1 + \xi) / 2 + p_m^+ \Pi(\xi, \omega) \exp [i(\tau + \varphi_p(\omega))], \]  
(19)

where \( \Pi(\xi, \omega) = \rho \nu \alpha \beta b (\xi^2 - 1) / 2 \left( 12 \gamma B_p + 2 \varepsilon^2 \alpha A_p \right)^2 + (12 \gamma A_p - 2 \varepsilon^2 \alpha B_p)^2 \right)^{1/2}, \tg \varphi_p(\omega) = (2 \varepsilon^2 \alpha B_p - 12 \gamma A_p) / (12 \gamma B_p + 2 \varepsilon^2 \alpha A_p).

The first component in the law of the channel wall shift (18) is the shift under static pressure, and the second component represents the shift, determined by the dynamic pressure in the channel. In the expression for dynamic pressure (19) the component \( p^+(\tau)(1 + \xi) / 2 \) reflects linear pressure fall along the channel, and the second component represents the pressure in the liquid due to its squeeze by elastically fixed channel wall. It is clearly seen, that the first component of the dynamic pressure along the channel is not larger than the suggested pressure \( p^+(\tau) \) at the butt end.

Let us notice, that in fact, the amplitude frequency characteristic \( A(\omega) \) and phase frequency characteristic \( \varphi_p(\omega) \) of elastically fixed channel wall on the suggested pressure pulsating at the butt end are received in (18). The function \( \Pi(\xi, \omega) \), which can be considered as frequency-dependent distribution function
of relative amplitudes of pressure along the channel, is received in \((19)\). In analogy, the function \(\varphi_p(\omega)\) can be considered frequency-dependent function of phase pressure shift in the channel in relation to the original disturbance at the butt end. In the case of the fixed value of longitudinal coordinate \(\xi\), the suggested function \(\Pi(\xi, \omega)\) presents amplitude frequency pressure characteristic in the cross-section of channel.

Conclusions

It is known, that the amplitude of elastic construction fluctuation shifts are able to exceed disturbing action by some orders and more \([1, 3, 4, 11, 12, 13]\). That is why, under resonance oscillations frequencies, it is quite possible to force a substantial growth of elastically fixed channel wall fluctuations and the same growth of the amplitude of liquid dynamic pressure, due to its squeeze by elastically fixed channel wall, as well. Taking into account the harmonic character of pressure change in the course of time, it is possible to confirm, that under resonance frequencies the liquid pressure in channel will become smaller than the one of the saturated vapor. Therefore, even slight pressure pulsating at the butt ends, can provoke significant pressure changes in the channel, due to liquid squeeze by elastically fixed channel wall under resonance oscillations frequencies. It well leads to the emergence of vibration cavitations in the liquid. It is quite obvious, that for the law of elastically fixed channel wall shifts and hydrodynamic parameters of viscous incompressible liquid layer can get practical implementation for defining resonance oscillations frequencies, corresponding to the conditions of cavitations emergence. The expressions, mention above, can be of use for evaluation the possibilities of resonance oscillations formation, on the basis of the suggested frequency range of possible pressure pulsating at the channel butt end.

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