A Note on Application of
the \( \coth_a(\xi) \) Expansion Method
to the Davey–Stewartson Equation

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Abstract

The recent paper "The \( \coth_a(\xi) \) expansion method and its application to the Davey–Stewartson equation" (Applied Mathematical Sciences 8 (2014) 3851–3868) is analyzed. We show that the authors solved equations with the well known general solutions. The two first solutions of paper correspond to the equation for the Weierstrass elliptic function and the two other solutions can be obtained from the general solution for the second-order linear ordinary differential equation for free oscillations.

Keywords: Davey–Stewartson equation, Ordinary differential equation, General solution, Exact solution, Weierstrass elliptic function

In the recent paper Bashir and Moussa (2014) [1] considered the Davey–Stewartson nonlinear partial differential equations in the form

\[
\begin{align*}
  iW_t + \delta_0 W_{xx} + W_{yy} &= \delta_1 |W|^2 W + \delta_2 W F_x, \\
  F_{xx} + \delta_3 F_{yy} &= (|W|^2)_x. 
\end{align*}
\]
The authors choose the following traveling wave transformation

\[ W = u(\xi)e^{i\vartheta}, \quad F(\xi) = v(\xi), \]

\[ \xi = k_1 x + k_2 y - 2(k_2 l_2 + \delta_0 k_1 l_1) t, \]

\[ \vartheta = l_1 x + l_2 y + l_3 t, \]

where \( k_1, k_2, l_1, l_2 \) and \( l_3 \) are arbitrary constants. Equations (1) become

\[ (l_3 + \delta_0 l_1^2 + l_2^2) u + \delta_1 u^3 - (\delta_0 k_1^2 + k_2^2) u'' + k_1 \delta_2 u v' = 0, \]

\[ (k_1^2 + \delta_3 k_2^2) v'' - 2k_1 u u' = 0. \]

Integrating the second equation in (3) with respect to \( \xi \) and setting the constant of integration to zero they found

\[ v' = \frac{k_1}{(k_1^2 + \delta_3 k_2^2)} u^2. \]

Substituting (4) into (3) the authors obtained the following second-order autonomous differential equation:

\[ \left( \delta_1 + \frac{\delta_3 k_1^2}{k_1^2 + \delta_3 k_2^2} \right) u^3 + (l_3 + \delta_0 l_1^2 + l_2^2) u - (\delta_0 k_1^2 + k_2^2) u'' = 0. \]

In fact, taking arbitrary constant into account we have the following equation:

\[ \left( \delta_1 + \frac{\delta_3 k_1^2}{k_1^2 + \delta_3 k_2^2} \right) u^3 + (l_3 + \delta_0 l_1^2 + l_2^2 + C_1 k_1 \delta_2) u - (\delta_0 k_1^2 + k_2^2) u'' = 0, \]

where \( C_1 \) is an arbitrary constant. By means of the \( \text{coth}_u(\xi) \) expansion method the authors look for exact solutions of equation (5) (or (6)). However Eq.(5) (or (6)) has the well known general solution. Let us show this fact.

After multiplying by \( u' \) Eq.(6) can be integrated with respect to \( \xi \) again. In this case we have the equation in the form

\[ C_2 + \frac{1}{4} \left( \delta_1 + \frac{\delta_3 k_1^2}{k_1^2 + \delta_3 k_2^2} \right) u^4 + \frac{1}{2} \left( l_3 + \delta_0 l_1^2 + l_2^2 + C_1 k_1 \delta_2 \right) u^2 - \frac{1}{2} \left( \delta_0 k_1^2 + k_2^2 \right) (u')^2 = 0. \]

where \( C_1, C_2 \) are arbitrary constants. Using change of variables \( u^2 = v \) we finally obtain the first-order differential equation

\[ (v')^2 = \frac{2(\delta_1 + \frac{\delta_3 k_1^2}{k_1^2 + \delta_3 k_2^2})}{\delta_0 k_1^2 + k_2^2} v^3 + \frac{4(l_3 + \delta_0 l_1^2 + l_2^2 + C_1 k_1 \delta_2)}{\delta_0 k_1^2 + k_2^2} v^2 + \frac{8C_2}{\delta_0 k_1^2 + k_2^2} v. \]
To simplify our further calculations we set
\[
(v')^2 = D_1 v^3 + D_2 v^2 + D_3 v,
\]
\[
D_1 = \frac{2(\delta_1 + \frac{\delta_2 k_1^2}{k_1^2 + \delta_3 k_2^2})}{\delta_0 k_1^2 + k_2^2},
\]
\[
D_2 = \frac{4(l_3 + \delta_0 l_1^2 + l_2^2 + C_1 k_1 \delta_2)}{\delta_0 k_1^2 + k_2^2},
\]
\[
D_3 = \frac{8 C_2}{\delta_0 k_1^2 + k_2^2}.
\]

Eq. (9) is well known equation for the Weierstrass elliptic function. This equation passes the Painlevé test and therefore it is integrable by the inverse scattering transform. Its general solution was found more than one century ago and may be expressed via the Weierstrass elliptic function (see, for example, Polyanin and Zaitsev [3]). We can see it if we substitute \( v = \frac{4}{D_1} \wp(\xi) - \frac{D_2}{3 D_1} \) into Eq. (9). In this case we obtain the following equation for the Weierstrass elliptic function
\[
(v')^2 = 4 \wp^3 - g_2 \wp - g_3,
\]
\[
g_2 = -\frac{1}{4} D_1 D_3 + \frac{1}{12} D_2^2, \quad g_3 = \frac{1}{432} D_2 (9 D_1 D_3 - 2 D_2^2).
\]

As a result we have a solution of Eq. (7) in the form
\[
(u(\xi))^2 = \frac{4}{D_1} \wp(\xi - \xi_0; g_2; g_3) - \frac{D_2}{3 D_1},
\]
where
\[
g_2 = -\frac{1}{4} D_1 D_3 + \frac{1}{12} D_2^2, \quad g_3 = \frac{1}{432} D_2 (9 D_1 D_3 - 2 D_2^2).
\]

We checked the two first exact solutions by Bashir and Moussa [1] and found that this solutions can be obtained from (11). Let us show this fact. Assuming the following substitution:
\[
C_1 = 0, \quad l_3 = -2 k^2 \ln^2(a)(\delta_0 k_1^2 + k_2^2) - \delta_0 l_1^2 - l_2^2,
\]
\[
C_2 = \frac{(k_1^2 + \delta_3 k_2^2)(\delta_0 k_1^2 + k_2^2)^2}{(\delta_1 + \delta_2) k_1^2 + \delta_1 \delta_3 k_2^2} k^4 \ln^4(a),
\]
we get
\[
u(\xi)^2 = 2 \frac{(k_1^2 + \delta_3 k_2^2)(\delta_0 k_1^2 + k_2^2)}{(\delta_1 + \delta_2) k_1^2 + \delta_1 \delta_3 k_2^2} k^2 \ln^2(a) \left[ -\frac{2}{3} + \tan^2 \left( k \ln(a)(\xi - \xi_0) \right) \right] +
\]
\[
+ \frac{4}{3} k^2 \ln^2(a) \frac{(k_1^2 + \delta_3 k_2^2)(\delta_0 k_1^2 + k_2^2)}{(\delta_1 + \delta_2) k_1^2 + \delta_1 \delta_3 k_2^2}.
\]
After simplification, we have

\[ u(\xi)^2 = 2 \frac{(k_1^2 + \delta_3 k_2^2)(\delta_0 k_1^2 + k_2^2)}{((\delta_1 + \delta_2)k_1^2 + \delta_1 \delta_3 k_2^2)} k^2 \ln^2(a) \tanh^2(k \ln(a)(\xi - \xi_0)) \] (15)

and using the expression for \(\tanh(\xi)\) we finally obtain

\[ u(\xi) = \pm \sqrt{2 \frac{(k_1^2 + \delta_3 k_2^2)(\delta_0 k_1^2 + k_2^2)}{((\delta_1 + \delta_2)k_1^2 + \delta_1 \delta_3 k_2^2)} k^2 \ln(a) \left( p e^{k \ln(a)\xi} + q e^{-k \ln(a)\xi} \right) / \left( p e^{k \ln(a)\xi} - q e^{-k \ln(a)\xi} \right)}, \] (16)

where \( p = e^{-k \ln(a)\xi_0} \) and \( q = -e^{k \ln(a)\xi_0} \). So, we reduced solution (11) to the first solution (see formula (23)) of paper by Bashir and Moussa [1]. The second solution (see formula (30)) from this paper we can obtain if we identify \( p \) and \( q \) by another way:

\[ p = e^{-k \ln(a)\xi_0}, \quad q = e^{k \ln(a)\xi_0}. \]

Solutions (37) and (44) from [1] were obtained for another case of equation (5) (or (6), (7)):

\[ (l_3 + \delta_0 l_1^2 + l_2^2) u - (\delta_0 k_1^2 + k_2^2) u'' = 0. \] (17)

The term with \( u^3 \) is absent as \( k_2 = \sqrt{-\frac{\delta_1 + \delta_2}{\delta_1 \delta_3}} k_1 \) and consequently \( D_1 = 0 \). Eq.(17) is the second-order linear ordinary differential equation for free oscillations. It has the well known general solution (see, for example, [3]) and that’s why solutions (37) and (44) by Bashir and Moussa [1] can be obtained from this general solution.

In conclusion, it should be noted that the question about solution of equations (9),(17) was discussed in paper [2].

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References


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