Deductive Systems of $BL$-Algebras

Based on Soft Set Theory

Kyoung Ja Lee

Department of Mathematics Education
Hannam University
Daejeon 306-791, Korea

Young Bae Jun

Department of Mathematics Education
Gyeongsang National University
Jinju 660-701, Korea

Seok-Zun Song*

Department of Mathematics
Jeju National University
Jeju 690-756, Korea

Copyright © 2014 Kyoung Ja Lee, Young Bae Jun and Seok-Zun Song. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

The notion of an int-soft deductive system of a $BL$-algebra is introduced, and several properties are investigated. Characterizations of an int-soft deductive system in a $BL$-algebra are established. Given a soft set $\mathcal{F}_L$, the least int-soft deductive system of a $BL$-algebra $L$ which contains $\mathcal{F}_L$ is constructed.

Mathematics Subject Classification: 06F35, 03G25, 06D72

*Corresponding author.
Keywords: BL-algebra, Deductive system, Int-soft deductive system, Inclusive deductive system

1 Introduction

Fuzzy logic grows as a new discipline from the necessity to deal with vague data and imprecise information caused by the indistinguishability of objects in certain experimental environments. As mathematical tools fuzzy logic is only using $[0, 1]$-valued maps and certain binary operations $*$ on the real unit interval $[0, 1]$ known also as left-continuous $t$-norms. It took sometime to understand partially ordered monoids of the form $([0, 1], \leq, *)$ as algebras for $[0, 1]$-valued interpretations of a certain type of non-classical logic-the so-called monoidal logic. BL-algebras, which have been introduced by Hájek as algebraic structures of Basic Logic, arise naturally in the analysis of the proof theory of propositional fuzzy logics. In [5], Ko and Kim investigated some properties of BL-algebras, and they [6] also studied relationships between closure operators and BL-algebras. Jun and Ko [3] showed that a deductive system of a BL-algebra can be represented as a union of special sets, and gave a characterization of a deductive system of a BL-algebra. They also discussed how to generate a deductive system by a set, and investigated some related properties. Jun et al. [4] introduced special sets in a BL-algebra, and gave an example in which a special set is not a deductive system. They stated conditions for these special sets to be a deductive system. Using this special set, they provided an equivalent condition of a deductive system. They proved that a deductive system can be represented by the union of such special sets. They also introduce the notion of commutative BL-algebras, and gave its characterizations. They discussed the notion of sensitive deductive systems, and gave some conditions for a deductive system to be a sensitive deductive system.

In this paper, using the soft set theory, we introduce the notion of int-soft deductive system of a BL-algebra, and investigate several properties. We establish characterizations of an int-soft deductive system in a BL-algebra. Given a soft set $F_L$, we build the least int-soft deductive system of a BL-algebra $L$ which contains $F_L$.

2 Preliminary Notes

2.1 Basic results on BL-algebras

Definition 2.1 ([9]) A BL-algebra is an algebra $(L, \land, \lor, \otimes, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ that satisfies the following conditions:

(A1) $(L, \land, \lor, 0, 1)$ is a bounded lattice,
(A2) \((L, \odot, 1)\) is a commutative monoid,

(A3) \(\odot\) and \(\to\) form an adjoint pair, i.e., \(z \leq x \to y\) if and only if \(x \odot z \leq y\) for all \(x, y, z \in L\), where \(\leq\) is the lattice ordering on \(L\),

(A4) \((\forall x, y \in L)\) \((x \land y = x \odot (x \to y))\),

(A5) \((\forall x, y \in L)\) \(((x \to y) \lor (y \to x) = 1)\).

**Lemma 2.2** ([2, 9]) Let \(L\) be a BL-algebra. The following properties hold:

1. \((\forall x, y \in L)\) \((x \leq y \iff x \to y = 1)\),
2. \((\forall x, y, z \in L)\) \((x \to (y \to z) = (x \odot y) \to z)\),
3. \((\forall x, y \in L)\) \((x \odot y \leq x, y)\),
4. \((\forall x, y \in L)\) \((x \odot y \leq x \land y)\),
5. \((\forall x, y \in L)\) \((x \odot y \leq x \to y)\),
6. \((\forall x, y, z \in L)\) \(((x \to y) \odot (y \to z) \leq x \to z)\),
7. \((\forall x, y \in L)\) \((x \lor y = ((x \to y) \to y) \land ((y \to x) \to x))\),
8. \((\forall x, y, z \in L)\) \((x \to (y \to z) = y \to (x \to z))\),
9. \((\forall x, y, z \in L)\) \((x \to y \leq (z \to x) \to (z \to y))\),
10. \((\forall x, y \in L)\) \((x \leq y \iff z \to x \leq z \to y, y \to z \leq x \to z)\),
11. \((\forall x, y, z \in L)\) \(((x \to y) \to (x \to z) \leq x \to (y \to z))\).

**Definition 2.3** ([8]) A nonempty subset \(F\) of \(L\) is called a deductive system of \(L\) if it satisfies

(i) \(1 \in F\),

(ii) \((\forall x \in F)\) \((\forall y \in L)\) \((x \to y \in F \Rightarrow y \in F)\).

**Lemma 2.4** ([8]) Let \(F\) be a nonempty subset of \(L\). Then \(F\) is a deductive system of \(L\) if and only if it satisfies

1. \((\forall x \in F)\) \((\forall y \in L)\) \((x \leq y \Rightarrow y \in F)\).
2. \((\forall x, y \in L)\) \((x, y \in F \Rightarrow x \odot y \in F)\).
2.2 Basic results on soft set theory

Soft set theory was introduced by Molodtsov [7] and Çağman et al. [1].

In what follows, let \( U \) be an initial universe set and \( E \) be a set of parameters. We say that the pair \((U, E)\) is a soft universe. Let \( \mathcal{P}(U) \) (resp. \( \mathcal{P}(E) \)) denotes the power set of \( U \) (resp. \( E \)).

By analogy with fuzzy set theory, the notion of soft set is defined as follows:

**Definition 2.5 ([1, 7])** A soft set of \( E \) over \( U \) (a soft set of \( E \) for short) is:

any function \( f_A : E \to \mathcal{P}(U) \), such that \( f_A(x) = \emptyset \) if \( x \notin A \), for \( A \in \mathcal{P}(E) \),

or, equivalently, any set

\[ \mathcal{F}_A := \{(x, f_A(x)) \mid x \in E, f_A(x) \in \mathcal{P}(U), f_A(x) = \emptyset \text{ if } x \notin A\}, \]

for \( A \in \mathcal{P}(E) \).

**Definition 2.6 ([1])** Let \( \mathcal{F}_A \) and \( \mathcal{F}_B \) be soft sets of \( E \). We say that \( \mathcal{F}_A \) is a soft subset of \( \mathcal{F}_B \), denoted by \( \mathcal{F}_A \subseteq \mathcal{F}_B \), if \( f_A(x) \subseteq f_B(x) \) for all \( x \in E \).

3 Int-soft deductive systems

In what follows, denote by \( S(U, L) \) the set of all soft sets of \( L \) over \( U \) where \( L \) is a \( BL\)-algebra unless otherwise specified.

**Definition 3.1** A soft set \( \mathcal{F}_L \in S(U, L) \) is called an int-soft deductive system of \( L \) if it satisfies:

\[ (\forall \gamma \in \mathcal{P}(U)) (\mathcal{F}_L^\gamma \neq \emptyset \implies \mathcal{F}_L^\gamma \text{ is a deductive system of } L), \]

where \( \mathcal{F}_L^\gamma = \{x \in L \mid \gamma \subseteq f_L(x)\} \) which is called the \( \gamma \)-inclusive set of \( \mathcal{F}_L \).

If \( \mathcal{F}_L \) is an int-soft deductive system of \( L \), every \( \gamma \)-inclusive set \( \mathcal{F}_L^\gamma \) is called an inclusive deductive system of \( L \).

**Example 3.2** Let \( L = \{0, a, b, 1\} \) and \( 0 < a < b < 1 \). For all \( x, y \in L \), we define \( x \land y = \min\{x, y\}, x \lor y = \max\{x, y\}, \text{ and } \odot \text{ and } \rightarrow \text{ as follows:} \)

\[
\begin{array}{c|cccc}
\odot & 0 & a & b & 1 \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & a & a & a \\
b & 0 & a & b & b \\
1 & 0 & a & b & 1 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\rightarrow & 0 & a & b & 1 \\
0 & 1 & 1 & 1 & 1 \\
a & a & 1 & 1 & 1 \\
b & b & 0 & a & 1 \\
1 & 0 & a & b & 1 \\
\end{array}
\]
Let $\mathcal{F}_L \in S(U, L)$ be given as follows:

$$\mathcal{F}_L = \{(0, \gamma_3), (a, \gamma_3), (b, \gamma_2), (1, \gamma_1)\}$$

where $\gamma_1, \gamma_2$ and $\gamma_3$ are subsets of $U$ with $\gamma_1 \supseteq \gamma_2 \supseteq \gamma_3$. Then $\mathcal{F}_L$ is an int-soft deductive system of $L$.

**Example 3.3** Let $L = \{0, a, b, c, d, 1\}$ be a set with Hasse diagram and Cayley tables as follows:

<table>
<thead>
<tr>
<th></th>
<th>$\circ$</th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>1</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>1</td>
<td>1</td>
<td>c</td>
<td>c</td>
<td>1</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>b</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

For every $x, y \in L$, define $x \wedge y = x \circ (x \rightarrow y)$ and

$$x \vee y = ((x \rightarrow y) \rightarrow y) \circ (((x \rightarrow y) \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x)).$$

Then $(L; \leq, \wedge, \vee, \circ, \rightarrow, 0, 1)$ is a BL-algebra (cf. [4]). Let $\mathcal{F}_L \in S(U, L)$ be given as follows:

$$\mathcal{F}_L = \{(1, \gamma_1), (a, \gamma_1), (b, \gamma_1), (c, \gamma_2), (d, \gamma_2), (0, \gamma_2)\}$$

where $\gamma_1$ and $\gamma_2$ are subsets of $U$ with $\gamma_1 \supseteq \gamma_2$. Then $\mathcal{F}_L$ is an int-soft deductive system of $L$.

We provide characterizations of a deductive system.

**Theorem 3.4** Let $\mathcal{F}_L \in S(U, L)$. Then $\mathcal{F}_L$ is an int-soft deductive system of $L$ if and only if the following assertions are valid:

$$\forall x \in L \left( f_L(x) \subseteq f_L(1) \right),$$

$$\forall x, y \in L \left( f_L(x \rightarrow y) \cap f_L(x) \subseteq f_L(y) \right).$$

**Proof.** Assume that $\mathcal{F}_L$ is an int-soft deductive system of $L$. For any $x \in L$, let $f_L(x) = \gamma$. Then $x \in \mathcal{F}_L^\gamma$. Since $\mathcal{F}_L^\gamma$ is a deductive system of $L$, we have $1 \in \mathcal{F}_L^\gamma$ and so $f_L(x) = \gamma \subseteq f_L(1)$. For any $x, y \in L$, let $f_L(x \rightarrow y) \cap f_L(x) = \gamma$. Then $x \rightarrow y \in \mathcal{F}_L^\gamma$ and $x \in \mathcal{F}_L^\gamma$. Since $\mathcal{F}_L^\gamma$ is a deductive system of $L$, it follows that $y \in \mathcal{F}_L^\gamma$. Hence $f_L(x \rightarrow y) \cap f_L(x) = \gamma \subseteq f_L(y)$.

Conversely, suppose that $\mathcal{F}_L$ satisfies two conditions (1) and (2). Let $\gamma \in \mathcal{P}(U)$ be such that $\mathcal{F}_L^\gamma \neq \emptyset$. Then there exists $a \in \mathcal{F}_L^\gamma$, and so $\gamma \subseteq f_L(a)$. It
follows from (1) that $\gamma \subseteq f_L(a) \subseteq f_L(1)$. Thus $1 \in \mathcal{F}_L$. Let $x, y \in L$ be such that $x \rightarrow y \in \mathcal{F}_L$ and $x \in \mathcal{F}_L$. Then $\gamma \subseteq f_L(x \rightarrow y)$ and $\gamma \subseteq f_L(x)$. It follows from (2) that

$$\gamma \subseteq f_L(x \rightarrow y) \cap f_L(x) \subseteq f_L(y),$$

that is, $y \in \mathcal{F}_L$. Thus $\mathcal{F}_L(\neq \emptyset)$ is a deductive system of $L$, and hence $\mathcal{F}_L$ is an int-soft deductive system of $L$.

**Theorem 3.5** Let $\mathcal{F}_L \in S(U, L)$. Then $\mathcal{F}_L$ is an int-soft deductive system of $L$ if and only if the following assertion is valid:

$$(\forall x, y, z \in L) (x \rightarrow (y \rightarrow z) = 1 \Rightarrow f_L(x) \cap f_L(y) \subseteq f_L(z)).$$

**Proof.** Assume that $\mathcal{F}_L$ is an int-soft deductive system of $L$ and let $x, y, z \in L$ be such that $x \rightarrow (y \rightarrow z) = 1$. It follows from (1) and (2) that

$$f_L(z) \supseteq f_L(y) \cap f_L(y \rightarrow z) \supseteq f_L(y) \cap (f_L(x) \cap f_L(x \rightarrow (y \rightarrow z)))$$

$$= f_L(y) \cap (f_L(x) \cap f_L(1)) = f_L(y) \cap f_L(x).$$

Conversely, suppose that $\mathcal{F}_L$ satisfies (3). Since $x \rightarrow (x \rightarrow 1) = 1$ for all $x \in L$, we have $f_L(x) = f_L(x) \cap f_L(x) \subseteq f_L(1)$ for all $x \in L$. Note that $(x \rightarrow y) \rightarrow (x \rightarrow y) = 1$ for all $x, y \in L$. Using (3), we have $f_L(x) \cap f_L(x \rightarrow y) \subseteq f_L(y)$ for all $x, y \in L$. It follows from Theorem 3.4 that $\mathcal{F}_L$ is an int-soft deductive system of $L$.

Note that $x \rightarrow (y \rightarrow z) = x \circ y \rightarrow z$ for all $x, y, z \in L$. Thus the following corollary follows from Theorem 3.5.

**Corollary 3.6** Let $\mathcal{F}_L \in S(U, L)$. Then $\mathcal{F}_L$ is an int-soft deductive system of $L$ if and only if the following assertion is valid:

$$(\forall x, y, z \in L) (x \circ y \rightarrow z = 1 \Rightarrow f_L(x) \cap f_L(y) \subseteq f_L(z)).$$

**Theorem 3.7** Let $\mathcal{F}_L \in S(U, L)$. Then $\mathcal{F}_L$ is an int-soft deductive system of $L$ if and only if the following assertions are valid:

$$\mathcal{F}_L \text{ is order-preserving, that is, if } x \leq y \text{ then } f_L(x) \subseteq f_L(y),$$

$$(\forall x, y \in L) (f_L(x) \cap f_L(y) \subseteq f_L(x \circ y)).$$

**Proof.** Let $\mathcal{F}_L$ be an int-soft deductive system of $L$ and let $x, y \in L$. If $x \leq y$, then $x \circ x \leq x \leq y$ and so $f_L(x) = f_L(x) \cap f_L(x) \subseteq f_L(y)$ by Lemma 2.2(1) and Corollary 3.6. Since $x \circ y \leq x \circ y$, it follows from Lemma 2.2(1) and Corollary 3.6 that $f_L(x) \cap f_L(y) \subseteq f_L(x \circ y)$. Conversely, assume that $\mathcal{F}_L$ satisfies two conditions (5) and (6). Let $x, y, z \in L$ be such that $x \circ y \rightarrow z = 1$, i.e., $x \circ y \leq z$. Then

$$f_L(x) \cap f_L(y) \subseteq f_L(x \circ y) \subseteq f_L(z).$$

Hence, by Corollary 3.6, we know that $\mathcal{F}_L$ is an int-soft deductive system of $L$. 

\[ \text{\textcopyright Kyung Ja Lee, Young Bae Jun and Seok-Zun Song} \]
Corollary 3.8 For any int-soft deductive system $\mathcal{F}_L$ of $L$, the following properties hold:

1. $(\forall x, y \in L) (f_L(x \rightarrow y) = f_L(1) \Rightarrow f_L(x) \subseteq f_L(y))$,
2. $(\forall x, y \in L) (f_L(x \odot y) = f_L(x) \cap f_L(y) = f_L(x \wedge y))$,
3. $(\forall x \in L) (\forall y \in \mathbb{N}) (f_L(x^n) = f_L(x))$, where $x^n = x \odot x \odot \cdots \odot x$ in which $x$ appears $n$-times for $n \in \mathbb{N}$,
4. $(\forall x \in L) (f_L(0) = f_L(x) \cap f_L(\neg x))$ where $\neg x = x \rightarrow 0$,
5. $(\forall x, y, z \in L) (f_L(x \rightarrow z) \supseteq f_L(x \rightarrow y) \cap f_L(y \rightarrow z))$,
6. $(\forall x, y, z \in L) (f_L(x \rightarrow y) \subseteq f_L(x \odot z \rightarrow y \odot z))$.

Proof. (1) Let $x, y \in L$ be such that $f_L(x \rightarrow y) = f_L(1)$. Then

$$f_L(x) = f_L(x) \cap f_L(1) = f_L(x) \cap f_L(x \rightarrow y) \subseteq f_L(y)$$

by (1) and (2).

(2) Lemma 2.2(3) and (5) imply that $f_L(x \odot y) \subseteq f_L(x) \cap f_L(y)$ for all $x, y \in L$. It follows from (6) that $f_L(x \odot y) = f_L(x) \cap f_L(y)$ for all $x, y \in L$. Lemma 2.2(4) and (5) imply that $f_L(x \odot y) \subseteq f_L(x \wedge y)$ for all $x, y \in L$. Obviously, $f_L(x \wedge y) \subseteq f_L(x) \cap f_L(y) = f_L(x \odot y)$, and thus $f_L(x \odot y) = f_L(x \wedge y)$ for all $x, y \in L$.

(3) It follows from (2).

(4) Note that $x \odot \neg x = 0$ for all $x \in L$. Using (2), we have

$$f_L(0) = f_L(x \odot \neg x) = f_L(x) \cap f_L(\neg x)$$

for all $x, y \in L$.

(5) follows from Lemma 2.2(6), (5) and (2).

(6) Since $x \rightarrow y \leq x \odot z \rightarrow y \odot z$ for all $x, y, z \in L$, it follows from (5) that $f_L(x \rightarrow y) \subseteq f_L(x \odot z \rightarrow y \odot z)$ for all $x, y, z \in L$.

Theorem 3.9 For any $\mathcal{F}_L \in S(U, L)$, the soft set $\mathcal{G}_L$ of $L$ with

$$g_L(x) = \bigcup \{f_L(a_1) \cap \cdots \cap f_L(a_n) \mid x \geq a_1 \odot \cdots \odot a_n, \ a_1, \cdots, a_n \in L\}$$

for all $x \in L$ is an int-soft deductive system of $L$ containing $\mathcal{F}_L$. Moreover if $\mathcal{H}_L$ is an int-soft deductive system of $L$ such that $\mathcal{F}_L \subseteq \mathcal{H}_L$, then $\mathcal{G}_L \subseteq \mathcal{H}_L$. 
Proof. Obviously, \( g_L(1) = f_L(1) \supseteq g_L(x) \) for all \( x \in L \). Let \( x, y \in L \) be such that
\[
x \geq a_1 \circ \cdots \circ a_n \quad \text{and} \quad x \rightarrow y \geq b_1 \circ \cdots \circ b_m
\]
for \( a_1, \ldots, a_n, b_1, \ldots, b_m \in L \). Then
\[
y \geq x \circ (x \rightarrow y) \geq a_1 \circ \cdots \circ a_n \circ b_1 \circ \cdots \circ b_m,
\]
and so \( g_L(y) \supseteq f_L(a_1) \cap \cdots \cap f_L(a_n) \cap f_L(b_1) \cap \cdots \cap f_L(b_m) \). On the other hand,
\[
g_L(x) \cap g_L(x \rightarrow y) = (\cup \{f_L(a_1) \cap \cdots \cap f_L(a_n) \mid x \geq a_1 \circ \cdots \circ a_n\})
\cap (\cup \{f_L(b_1) \cap \cdots \cap f_L(b_m) \mid x \rightarrow y \geq b_1 \circ \cdots \circ b_m\})
\]
\[
= \cup \{f_L(a_1) \cap \cdots \cap f_L(a_n) \cap f_L(b_1) \cap \cdots \cap f_L(b_m) \mid x \geq a_1 \circ \cdots \circ a_n,
\quad x \rightarrow y \geq b_1 \circ \cdots \circ b_m\}
\]
and thus \( g_L(x) \cap g_L(x \rightarrow y) \subseteq g_L(y) \). Therefore \( G_L \) is an int-soft deductive system of \( L \). Since \( x \geq x \circ x \) for all \( x \in L \), it follows from \( f_L(x) = f_L(x) \cap f_L(x) \subseteq g_L(x) \) for all \( x \in L \), that is \( F_L \subseteq G_L \). Now let \( H_L \) be an int-soft deductive system of \( L \) such that \( F_L \subseteq H_L \). Then for any \( x \in L \),
\[
g_L(x) = \cup \{f_L(a_1) \cap \cdots \cap f_L(a_n) \mid x \geq a_1 \circ \cdots \circ a_n, \ a_1, \ldots, a_n \in L\}
\subseteq \cup \{h_L(a_1) \cap \cdots \cap h_L(a_n) \mid x \geq a_1 \circ \cdots \circ a_n, \ a_1, \ldots, a_n \in L\} \subseteq h_L(x).
\]
Hence \( G_L \subseteq H_L \).

Theorem 3.9 shows that the int-soft deductive system \( G_L \) is the least int-soft deductive system of \( L \) that contains the given soft set \( F_L \). We say that \( G_L \) is the int-soft deductive system of \( L \) generated by \( F_L \), and is denoted by \( \langle F_L \rangle \) where
\[
\langle f_L \rangle(x) = \cup \{f_L(a_1) \cap \cdots \cap f_L(a_n) \mid x \geq a_1 \circ \cdots \circ a_n, \ a_1, \ldots, a_n \in L\}
\]
for all \( x \in L \).

The following example illustrates Theorem 3.9.

**Example 3.10** Let \( L \) be the BL-algebra in Example 3.2. Let \( F_L \in S(\mathbb{Z}, L) \) be given by
\[
F_L = \{(0, 12\mathbb{Z}), (a, 6\mathbb{Z}), (b, 6\mathbb{Z}), (1, 2\mathbb{Z})\}.
\]
One can easily check that the int-soft deductive system generated by \( F_L \) is
\[
\langle F_L \rangle = \{(0, 6\mathbb{Z}), (a, 6\mathbb{Z}), (b, 6\mathbb{Z}), (1, 2\mathbb{Z})\}.
\]

Let \( F_L \in S(U, L) \). For any \( a, b \in L \) and \( k \in \mathbb{N} \), we define
\[
F_L[a^k; b] := \{x \in L \mid f_L(a^k \rightarrow (b \rightarrow x)) = f_L(1)\} \quad (7)
\]
where \( f_L(a^k \to x) = f_L(a \to (a \to \cdots \to (a \to (a \to x)) \cdot \cdot \cdot ) \) in which \( a \) appears \( k \)-times. Note that 1, \( a, b \in F_L[a^k; b] \) for all \( a, b \in L \) and \( k \in \mathbb{N} \). Observe that \( F_L[0^m; a] = L = F_L[a^n; 0] \) for all \( a \in L \) and \( m, n \in \mathbb{N} \). Note that

\[
F_L[a^1; b] \subseteq F_L[a^2; b] \subseteq F_L[a^3; b] \subseteq \cdots
\]

for all \( a, b \in L \).

**Proposition 3.11** Let \( F_L \in S(U, L) \) satisfy (1) and \( f_L(x \to y) = f_L(x) \cup f_L(y) \) for all \( x, y \in L \). For any \( a, b \in L \) and \( k \in \mathbb{N} \), if \( x \in F_L[a^k; b] \), then \( y \to x \in F_L[a^k; b] \) for all \( y \in L \).

**Proof.** Assume that \( x \in F_L[a^k; b] \). Then \( f_L(a^k \to (b \to x)) = f_L(1) \), and so

\[
f_L(a^k \to (b \to (y \to x))) = f_L(a^k \to (y \to (b \to x))) = f_L(y \to (a^k \to (b \to x))) = f_L(y) \cup f_L(a^k \to (b \to x)) = f_L(y) \cup f_L(1) = f_L(1)
\]

for all \( y \in L \). Hence \( y \to x \in F_L[a^k; b] \) for all \( y \in L \).

**Proposition 3.12** For any \( F_L \in S(U, L) \), let \( a \in L \) satisfy the following assertion:

\[
(\forall x(\neq 0) \in L)(a \to x = 1).
\]

Then \( F_L[a^k; b] = L = F_L[b^k; a] \) for all \( b \in L \) and \( k \in \mathbb{N} \).

**Proof.** For any \( x \in L \), we have

\[
f_L(a^k \to (b \to x)) = f_L(a^{k-1} \to (a \to (b \to x))) = f_L(a^{k-1} \to (b \to (a \to x))) = f_L(a^{k-1} \to (b \to 1)) = f_L(1),
\]

and so \( x \in F_L[a^k; b] \). Similarly, \( x \in F_L[b^k; a] \).

**Proposition 3.13** Let \( F_L \in S(U, L) \) in which \( f_L \) is injective. If \( a \leq b \) in \( L \), then \( F_L[b^k; x] \subseteq F_L[a^k; x] \) for all \( x \in L \) and \( k \in \mathbb{N} \).

**Proof.** Let \( a, b, y \in L \) be such that \( a \leq b \). If \( y \in F_L[b^1; x] \), then \( f_L(b \to (x \to y)) = f_L(1) \). Since \( f_L \) is injective, it follows that \( b \to (x \to y) = 1 \), that is, \( b \leq x \to y \). Since \( a \leq b \), we have \( 1 = a \to b \leq a \to (x \to y) \) by Lemma 2.2(10). Hence \( a \to (x \to y) = 1 \) and so \( f_L(a \to (x \to y)) = f_L(1) \), which shows that \( y \in F_L[a^1; x] \). If \( y \in F_L[b^2; x] \), then

\[
f_L(b \to (b \to (x \to y))) = f_L(b^2 \to (x \to y)) = f_L(1).
\]
Thus $b \rightarrow (b \rightarrow (x \rightarrow y)) = 1$, i.e., $b \leq b \rightarrow (x \rightarrow y)$ since $f_L$ is injective.
Using (8) and (10) in Lemma 2.2, we have $1 = a \rightarrow b \leq a \rightarrow (b \rightarrow (x \rightarrow y))$ and so
\[ b \rightarrow (a \rightarrow (x \rightarrow y)) = a \rightarrow (b \rightarrow (x \rightarrow y)) = 1, \]
that is, $b \leq a \rightarrow (x \rightarrow y)$. It follows from Lemma 2.2(10) that
\[ 1 = a \rightarrow b \leq a \rightarrow (a \rightarrow (x \rightarrow y)) \]
and so that $a \rightarrow (a \rightarrow (x \rightarrow y)) = 1$. Thus
\[ f_L(a^2 \rightarrow (x \rightarrow y)) = f_L(a \rightarrow (a \rightarrow (x \rightarrow y))) = f_L(1), \]
that is, $y \in F_L[a^2; x]$. Continuing this process, we obtain that if $y \in F_L[b^k; x]$, then $y \in F_L[a^k; x]$ for all $x \in L$ and $k \in \mathbb{N}$. This completes the proof.

**Proposition 3.14** Let $F_L \in S(U, L)$ be order-preserving and satisfy (1). If $b \leq c$ in $L$, then $F_L[a^k; c] \subseteq F_L[a^k; b]$ for all $a \in L$ and $k \in \mathbb{N}$.

**Proof.** Let $a, b, c, d, e \in L$ be such that $b \leq c$. For any $k \in \mathbb{N}$, if $x \in F_L[a^k; c]$, then
\[ f_L(1) = f_L(a^k \rightarrow (c \rightarrow x)) = f_L(c \rightarrow (a^k \rightarrow x)) \]
\[ \subseteq f_L(b \rightarrow (a^k \rightarrow x)) = f_L(a^k \rightarrow (b \rightarrow x)) \]
by (8) and (10) in Lemma 2.2, and so $f_L(a^k \rightarrow (b \rightarrow x)) = f_L(1)$. Thus
\[ x \in F_L[a^k; b], \]
which completes the proof.

The following example shows that there exists $F_L \in S(U, L)$, $a, b \in L$ and $k \in \mathbb{N}$ such that $F_L[a^k; b]$ is not a deductive system of $L$.

**Example 3.15** Consider the BL-algebra $L = \{0, a, b, c, d, 1\}$ in Example 3.3. Let $F_L \in S(U, L)$ be given as follows:
\[ F_L = \{(0, \gamma_4), (a, \gamma_3), (b, \gamma_1), (c, \gamma_5), (d, \gamma_5), (1, \gamma_2)\} \]
where \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\} is a class of subsets of $U$ which is a poset with the following Hasse diagram:

```
\[ \gamma_5 \]
\[ \gamma_4 \]
\[ \gamma_3 \]
\[ \gamma_2 \]
\[ \gamma_1 \]
```

Then $F_L[a^2; b] = \{1, a, b\}$ is a deductive system of $L$, but $F_L[1^k; a] = \{1, a\}$ is not a deductive system of $L$ for all $k \in \mathbb{N}$.
We provide conditions for a set $F_L[a^k; b]$ to be a deductive system.

**Theorem 3.16** Let $F_L \in S(U, L)$ in which $f_L$ is injective. If $L$ satisfies the following identity:

$$\forall x, y \in L) (x \to (x \to y) = x \to y).$$

then $F_L[a^k; b]$ is a deductive system of $L$ for all $a, b \in L$ and $k \in \mathbb{N}$.

**Proof.** Assume that $L$ satisfies (9). Using (8), (9) and (10) in Lemma 2.2, we have

$$(x \to y) \to (x \to z) = (x \to y) \to (x \to (x \to z))$$

$$= x \to ((x \to y) \to (x \to z)) \geq x \to (y \to z)$$

for all $x, y, z \in L$. It follows from Lemma 2.2(11) that

$$(\forall x, y, z \in L) (x \to (y \to z) = (x \to y) \to (x \to z)).$$

Let $a, b, x, y \in L$ and $k \in \mathbb{N}$ be such that $x \to y \in F_L[a^k; b]$ and $x \in F_L[a^k; b]$. Then $f_L(a^k \to (b \to x)) = f_L(1)$ which implies that $a^k \to (b \to x) = 1$ since $f_L$ is injective. Using (10), we have

$$f_L(1) = f_L(a^k \to (b \to (x \to y)))$$

$$= f_L(a^{k-1} \to (a \to (b \to (x \to y))))$$

$$= f_L(a^{k-1} \to (a \to ((b \to x) \to (b \to y))))$$

$$= \cdots$$

$$= f_L((a^k \to (b \to x)) \to (a^k \to (b \to y)))$$

$$= f_L(1 \to (a^k \to (b \to y)))$$

$$= f_L(a^k \to (b \to y)),$$

which implies that $y \in F_L[a^k; b]$. Therefore $F_L[a^k; b]$ is a deductive system of $L$ for all $a, b \in L$ and $k \in \mathbb{N}$.

**Theorem 3.17** Let $F_L \in S(U, L)$ in which $f_L$ is injective. If $L$ satisfies the following inequality:

$$\forall x, y, z \in L) (x \to (y \to z) \leq (x \to y) \to (x \to z)).$$

then $F_L[a^k; b]$ is a deductive system of $L$ for all $a, b \in L$ and $k \in \mathbb{N}$.

**Proof.** Assume that $L$ satisfies (11). Then we have

$$(\forall x, y, z \in L) (x \to (y \to z) = (x \to y) \to (x \to z))$$

by Lemma 2.2(11). If we put $y = x$ and $z = y$ in (12), then

$$x \to (x \to y) = (x \to x) \to (x \to y) = 1 \to (x \to y) = x \to y.$$

It follows from Theorem 3.16 that $F_L[a^k; b]$ is a deductive system of $L$ for all $a, b \in L$ and $k \in \mathbb{N}$. 
\textbf{Theorem 3.18} Let $\mathcal{F}_L \in S(U, L)$ in which $f_L$ is injective. Then any subset $D$ of $L$ is a deductive system of $L$ if and only if the following assertion is valid:

$$\left( \forall a, b \in D \right) \left( \forall k \in \mathbb{N} \right) \left( \mathcal{F}_L[a^k; b] \subseteq D \right).$$

\textbf{Proof.} Assume that $D$ is a deductive system of $L$ and let $a, b \in L$ and $k \in \mathbb{N}$. If $x \in \mathcal{F}_L[a^k; b]$, then $f_L(a \rightarrow (a^{k-1} \rightarrow (b \rightarrow x))) = f_L(a^k \rightarrow (b \rightarrow x)) = f_L(1)$ and so $a \rightarrow (a^{k-1} \rightarrow (b \rightarrow x)) = 1 \in D$ since $f_L$ is injective. Since $D$ is a deductive system of $L$, it follows from Definition 2.3(ii) that $a^{k-1} \rightarrow (b \rightarrow x) \in D$. Continuing this process, we obtain $b \rightarrow x \in D$ and so $x \in D$. Therefore $\mathcal{F}_L[a^k; b] \subseteq D$ for all $a, b \in D$ and $k \in \mathbb{N}$.

Conversely, suppose that the condition (13) holds. Note that $1 \in \mathcal{F}_L[a^k; b] \subseteq D$. Let $x, y \in L$ be such that $x \rightarrow y \in D$ and $x \in D$. Then

$$f_L(x^k \rightarrow ((x \rightarrow y) \rightarrow y)) = f_L(x^{k-1} \rightarrow (x \rightarrow ((x \rightarrow y) \rightarrow y)))$$
$$= f_L(x^{k-1} \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow y))) = f_L(x^{k-1} \rightarrow 1) = f_L(1),$$

and thus $y \in \mathcal{F}_L[x^k; b] \subseteq D$ where $b = x \rightarrow y$. Therefore $D$ is a deductive system of $L$.

\textbf{Theorem 3.19} If $D$ is a deductive system of $L$, then

$$\left( \forall k \in \mathbb{N} \right) \left( D = \bigcup \{ \mathcal{F}_L[a^k; b] | a, b \in D \} \right).$$

\textbf{Proof.} Let $D$ be a deductive system of $L$. By Theorem 3.18, the inclusion

$$\bigcup \{ \mathcal{F}_L[a^k; b] | a, b \in D \} \subseteq D$$

holds. Let $x \in D$. Since $x \in \mathcal{F}_L[1^k; x]$ for all $k \in \mathbb{N}$, it follows that

$$D \subseteq \bigcup \{ \mathcal{F}_L[1^k; x] | x \in D \} \subseteq \bigcup \{ \mathcal{F}_L[a^k; b] | a, b \in D \}.$$ 

This completes the proof.

\textbf{Acknowledgements.} This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2012R1A1A2042193).

\textbf{References}


Received: September 9, 2014; Published: November 3, 2014