Instabilities in Reaction-Diffusion Systems

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Abstract

In this paper we discuss the solution stability of two-component reaction-diffusion systems with constant diffusion coefficients. Linear stability analysis is performed near the steady state solution of the system discussing the dependence of the system stability on its parameters. A comprehensive linear stability analysis results in three types of instabilities: (1) Stationary periodic, (2) Oscillatory uniform and (3) Stationary uniform. Precise parameter regimes are identified for each.

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1 Introduction

In reaction-diffusion systems, most studies have been devoted to patterns which emerge in excitable systems near a Hopf bifurcation [4] and to Turing structures arising from Turing instability[11, 10]. More recently, attention has turned towards patterns arising from the wave instability (see for example [2, 13]).

In the literature, the majority of researchers focus on specific two-component reaction-diffusion systems, considering one or two types of instabilities. To our knowledge, there is at present no general instability study in the literature and so the present paper provides a comprehensive study of the possible instabilities that arise in the two-species (say $U(x,t)$ and $V(x,t)$) system

$$\frac{\partial U}{\partial t} = D_u \nabla^2 U + f(U,V), \quad \frac{\partial V}{\partial t} = D_v \nabla^2 V + g(U,V),$$  \hspace{1cm} (1)
where $D_u$ and $D_v$ are the diffusion coefficients, $f$ and $g$ are the kinetics, which are the only nonlinear terms that appear in the system. Many reaction-diffusion models have been proposed and analysed, both mathematically and via numerical simulation. There are numerous possibilities for the exact form of the reaction kinetics, including the Gray-Scott model [8], the Gierer-Meinhardt model [6], the Schnackenberg model [3], the Brusselator model [15], and the Lengyel-Epstein model [9] (for overviews see for example [1, 10]). While each of the reference above tackles a specific reaction-diffusion system, by contrast, in this paper we perform a comprehensive linear instability analysis, that gives all the possible types of instabilities which arise in the above system with general reaction kinetics.

As a control parameter is varied continuously, a nonlinear reaction-diffusion system initially in a homogeneous steady state may undergo a bifurcation to form patterns. There are two kinds of instabilities (assuming that the perturbation, near a steady state of (1), is proportional to $e^{\sigma t + i k x}$, where $\sigma$ is the rate of growth or decay of perturbations, $k$ is the perturbation wave number): the first type is uniform instability, for which the most unstable wave number is $k = k_* = 0$, and the second type is a spatially periodic instability with $k = k_* \neq 0$. Depending on whether the eigenvalue $\sigma$ is real or complex, each of these two types has two subtypes: stationary when the eigenvalue is purely real and oscillatory in case of a complex eigenvalue. Thus there are four kinds of instabilities: (1) stationary uniform, (2) oscillatory uniform, (3) stationary periodic and (4) oscillatory periodic. Studies of reaction-diffusion systems have led to the characterization of three basic types of symmetry-breaking bifurcations: oscillatory uniform (Hopf), stationary periodic (Turing), and oscillatory periodic (wave), and these are responsible for the emergence of patterns.

The symmetry-breaking bifurcations are characterised as follows. The space-independent Hopf bifurcation breaks the temporal symmetry of the system and gives rise to oscillations that are uniform in space and periodic in time and in this case $Re(\sigma) = 0$ and $Im(\sigma) \neq 0$ at $k = 0$. The (stationary) Turing bifurcation breaks the spatial symmetry, leading to the formation of patterns that are stationary in time and oscillatory in space, and in this type $Re(\sigma) = 0$ and $Im(\sigma) = 0$ at $k = k_T \neq 0$. The wave bifurcation (oscillatory Turing or finite-wavelength Hopf) breaks both spatial and temporal symmetries, generating patterns that are oscillatory in space and time and in this type $Re(\sigma) = 0$ and $Im(\sigma) \neq 0$ at $k = k_w \neq 0$.

This paper consists of three sections. In the next section, canonical forms for the linearized system of (1) are introduced. Also, we derive the characteristic equations, and a formula for the critical wave number is deduced. In section 3, we discuss the instabilities when the diffusion ratio $D_v/D_u$ is less than unity, for two different cases, without loss of generality. A catalogue of the instabilities with conditions is presented in section 4.
2 Linear Stability Analysis

2.1 Linear system

Assume that system (1) has a spatially uniform steady state
\[ U(x,t) = U_0, \quad V(x,t) = V_0, \quad F(U_0,V_0) = 0, \quad G(U_0,V_0) = 0. \] (2)
The above uniform steady state solution can be either stable or unstable in re-
spect to disturbances. In order to examine under what conditions the spatially
homogeneous state becomes unstable with respect to small spatially inhomo-
genous fluctuations, we perturb the steady state \((U_0,V_0)\) with small pertur-
bations \((\hat{U}(x,t),\hat{V}(x,t))\) so that
\[ U(x,t) = U_0 + \hat{U}(x,t), \quad V(x,t) = V_0 + \hat{V}(x,t). \] (3)
We consider the problem in one spatial dimension \(\hat{x}\) and set \(\hat{t} = \hat{t}\). We
substitute from (3) into (1). Then using the fact that \((U_0,V_0)\) is a uniform
steady state solution and linearising the kinetics, \(f\) and \(g\), around the steady
state, gives
\[ \frac{\partial \hat{U}}{\partial \hat{t}} = D_u \frac{\partial^2 \hat{U}}{\partial \hat{x}^2} + a\hat{U} + b\hat{V}, \quad \frac{\partial \hat{V}}{\partial \hat{t}} = D_v \frac{\partial^2 \hat{V}}{\partial \hat{x}^2} + c\hat{U} + d\hat{V} \] (4)
where \(D_u, D_v, a, b, c,\) and \(d\) are all real numbers (the first two are non-negative;
the other four can take either sign) and
\[ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} \frac{\partial f}{\partial U} & \frac{\partial f}{\partial V} \\ \frac{\partial g}{\partial U} & \frac{\partial g}{\partial V} \end{array} \right)_{(U_0,V_0)}. \] (5)
The above linearized system, (4), provides the linear stability equations
determining the behaviour of small perturbations \((\hat{U},\hat{V})\) to the steady state
solution. To reduce the number of parameters appearing in these equations
to express then in a canonical form. Let the solution \((\hat{U},\hat{V})\) takes the form
\[ \hat{U} = e^{\alpha t}\hat{u}, \quad \hat{V} = e^{\alpha t}\hat{v}, \] and then substitute into the system (4)to obtain
\[ \frac{\partial \hat{u}}{\partial \hat{t}} = D_u \frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + b\hat{v}, \quad \frac{\partial \hat{v}}{\partial \hat{t}} = D_v \frac{\partial^2 \hat{v}}{\partial \hat{x}^2} + c\hat{u} + (d-a)\hat{v}. \] (6)
Now we substitute the following new variables
\[ u = \frac{\hat{u}}{b}, \quad v = \frac{\hat{v}}{|d-a|}, \quad x = \sqrt{\frac{|d-a|}{D_u}} \hat{x}, \quad t = |d-a|\hat{t}. \] (7)
so that (6) becomes
\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + v, \quad \frac{\partial v}{\partial t} = \lambda \frac{\partial^2 v}{\partial x^2} + \alpha u \pm v, \tag{8}
\]
where
\[
\lambda = \frac{D_v}{D_u}, \quad \alpha = \frac{bc}{(d - a)^2}. \tag{9}
\]
The ± sign in the second equation in (8) results in two separate cases: the positive sign when \( d - a > 0 \), and the negative sign when \( d - a < 0 \). Each case will be studied separately in detail.

### 2.2 The characteristic equation and wave number

Here we derive the characteristic equation, then the eigenvalues of the two systems displayed in (8). Also, we derive a formula that gives a critical wave number. As the coefficients involved with the systems displayed in equations (8) are constants, we assume the solution of these systems in the Fourier form, i.e.,
\[
\begin{pmatrix} u \\ v \end{pmatrix} \propto e^{\sigma t + ikx}, \tag{10}\]
where \( \sigma \) is the rate of growth or decay of perturbations, \( k \) is the perturbation wave number. Thus we can say that the characteristic equation can be written as
\[
\det [R - k^2 D - \sigma I] = 0, \tag{11}\]
where \( I \) is the identity matrix, and the matrix \( R - k^2 D \) is shown in table 1. Hence the eigenvalues are the roots of the equation
\[
\sigma^2 - \text{Tr}\sigma + \Delta = 0, \tag{12}\]
where \( \text{Tr} \) and \( \Delta \) are the trace and the determinant of the matrix \( R - k^2 D \), respectively. We are interested in the root with the largest real part, \( \sigma^+ \), which can be written as
\[
\sigma^+ = \frac{1}{2} \left( \text{Tr} + \sqrt{\text{Tr}^2 - 4\Delta} \right). \tag{13}\]
We determine the maximum of the real part of this root and the corresponding wave number \( k \) for each of the above two systems shown in table 1, separately. The results for the system (4) are obtained. We undo the substitutions using (7) to find the larger eigenvalue, \( \sigma^+ \) and the wave number \( k \), the growth rate...
and wave number corresponding to (4), which can be written as

\[ \sigma^+ = a + |d - a|\sigma^+, \quad k = \sqrt{\frac{|d - a|}{D_u}} k, \]  

(14)

where \( \sigma^+ \) is displayed in (13), the characteristic wave number. We examine the

Table 1: Two different canonical forms of the linearised system of (1) and the corresponding characteristic matrix.

<table>
<thead>
<tr>
<th>Case</th>
<th>Equations</th>
<th>Conditions</th>
<th>( \mathbf{R} - k^2\mathbf{D} )</th>
</tr>
</thead>
</table>
| I    | \[ \begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + v \\
\frac{\partial v}{\partial t} &= \lambda \frac{\partial^2 v}{\partial x^2} + \alpha u + v
\end{align*} \] | \( d - a > 0 \) | \( \begin{pmatrix} -k^2 & 1 \\ \alpha & 1 - \lambda k^2 \end{pmatrix} \) |
| II   | \[ \begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + v \\
\frac{\partial v}{\partial t} &= \lambda \frac{\partial^2 v}{\partial x^2} + \alpha u - v
\end{align*} \] | \( d - a < 0 \) | \( \begin{pmatrix} -k^2 & 1 \\ \alpha & -1 - \lambda k^2 \end{pmatrix} \) |

maximum value of the temporal growth rate and the associated wave number. From (13), one can deduce that \((\partial \sigma^+ / \partial k)_{k=0} = 0\), hence at zero wave number \( \sigma^+(0) \) can be a maximal value. In our analysis we compare \( \sigma^+(0) \) with other maxima of the eigenvalue if exist at \( k = k_* \neq 0 \), to find the most positive (least negative) value. To find the possible value of \( k_* \), we differentiate (12) with respect to \( k^2 \) and rearrange (using (13)) to obtain

\[ \frac{\partial \sigma^+}{\partial k^2} = \frac{\sigma^+(\partial \text{Tr} / \partial k^2) - (\partial \Delta / \partial k^2)}{\sqrt{\text{Tr}^2 - 4\Delta}}, \]  

(15)

then we set the condition \( \frac{\partial \sigma^+}{\partial k^2} = 0 \), to obtain

\[ \left[ \sigma^+(\partial \text{Tr} / \partial k^2) - (\partial \Delta / \partial k^2) \right]_{k=k_*} = 0, \]  

(16)

then we substitute \( \sigma^+ \) from (13) yields

\[ \left[ \left( \text{Tr} + \sqrt{\text{Tr}^2 - 4\Delta} \right) \frac{\partial \text{Tr}}{\partial k^2} - 2 \frac{\partial \Delta}{\partial k^2} \right]_{k=k_*} = 0, \]  

(17)

which can be solved for a positive value of \( k_* \).
3 Systems with Diffusion Ratio $0 \leq \lambda \leq 1$

In this section we discuss the bifurcation types in case of $D_v \leq D_u$ ($\lambda \leq 1$), to the two cases shown in (1). It is obvious that results for the case $D_v > D_u$, i.e. $\lambda > 1$, can be obtained from results for $\lambda < 1$ by swapping $u$ and $v$, also interchange the reaction terms $f$ and $g$.

3.1 Case I

In case I (refer to table 1), we substitute $Tr = 1 - (1 + \lambda)k^2$ and $\Delta = \lambda k^4 - k^2 - \alpha$ into (13) to obtain

$$\sigma_I^+ = \frac{1}{2} \left[ 1 - (1 + \lambda)k^2 + \sqrt{[1 + (1 - \lambda)k^2]^2 + 4\alpha} \right],$$

and into (17), then we rearrange to obtain a quadratic in $k^2$ which can be solved for $k^*$ to obtain

$$k^*_2 = \frac{1}{1 - \lambda} \left[ -1 + \sqrt{-\alpha(1 + \lambda)^2} \right],$$

provided that $\lambda + \alpha(1 + \lambda)^2 < 0$. Hence $k_*$ exists in all regions in $\alpha, \lambda$ space plot displayed in figure 1, except region R1 (where $\lambda + \alpha(1 + \lambda)^2 > 0$), and constant $k_*$ contours are plotted and shown in the figure 1. Now we discuss the existence of maximum values of the growth rate, the real part of $\sigma_I^+$. When $\alpha \geq -1/4$ (the two regions R1 and R2 figure 1), the eigenvalue $\sigma_I^+$ is purely real. In region R2, where $-1/4 \leq \alpha < -\lambda/(1 + \lambda)^2$, the real eigenvalue $\sigma_I^+$ has a maximum, which is given by (from (16))

$$(\sigma_I^m)_m = \frac{1 - 2\lambda k^2_*}{1 + \lambda},$$

where $k_*$ is displayed in (19), and in this region stationary periodic instability can exist. However, in region R1 where $\alpha \geq -\lambda/(1 + \lambda)^2$, $k_*$ does not exist, and we can say that the maximum of $\sigma_I^+$ occurs only at $k = 0$. Thus the maximum is $\sigma_I^+(0) = [1 + \sqrt{(1 + 4\alpha)}]/2$, and the instability is of stationary uniform type. When $\alpha < -1/4$, regions R3 and R4, $\sigma_I^+$ is complex in the range $0 \leq k^2 < (1 + 2\sqrt{-\alpha})/(1 - \lambda)$, and real when $k^2 \geq (1 + 2\sqrt{-\alpha})/(1 - \lambda)$. There are two maxima for $\sigma_I^+$, the first occurs at $k = 0$ and equals $1/2$ which is the real part of $\sigma_I^+(0)$. The second maximum which is given by (20), occurs $k = k_* \neq 0$ (shown in (19)) and in this case the eigenvalue is pure real. We examine which one of the two maxima is larger, by checking the difference .

As $(1 - 2\lambda k^2_*)/(1 + \lambda) - (1/2) = (1 + \lambda - 4\sqrt{-\alpha\lambda})/(2(1 - \lambda))$, hence the second
is greater than the first when $1 + \lambda - 4\sqrt{-\alpha \lambda} > 0$ (or $(1 + \lambda)^2 + 16\alpha \lambda > 0$). Hence the type of instability is stationary periodic when $\alpha < -\frac{1}{4}$ and $(1 + \lambda)^2 + 16\alpha \lambda > 0$ (region R3). However when $(1 + \lambda)^2 + 16\alpha \lambda < 0$ (in region R4), the type is oscillatory uniform. On the border between these two regions (regions R3 and R4, the curve 2 in figure 1), where $(1 + \lambda)^2 + 16\alpha \lambda = 0$, the two types of instability co-exist. From the above discussion, the maximum growth rate $Re(\sigma^+_I)m$ can be written as

$$Re(\sigma^+_I)m = \begin{cases} \frac{(1 + \sqrt{1 + 4\alpha})}{2} & \alpha \geq -\frac{\lambda}{(1 + \lambda)^2}, \quad Im(\sigma^+_I) = 0, \\ \frac{1 - 2\lambda k^2}{1 + \lambda} & -\frac{(1 + \lambda)^2}{16\lambda} \leq \alpha < -\frac{\lambda}{(1 + \lambda)^2}, \quad Im(\sigma^+_I) = 0, \\ \frac{1}{2} & \alpha < -\frac{(1 + \lambda)^2}{16\lambda}, \quad Im(\sigma^+_I) = \sqrt{|\alpha + 1/4|}, \end{cases}$$

(21)

where $k_*$ is displayed in (19).

### 3.2 Case II

In case II, we substitute $Tr = -1 - (1 + \lambda)k^2$ and $\Delta = \lambda k^4 + k^2 - \alpha$ into (13) to obtain

$$\sigma^+_{II} = \frac{1}{2} \left[ -1 - (1 + \lambda)k^2 + \sqrt{((1 - \lambda)k^2 - 1)^2 + 4\alpha} \right],$$

(22)
Hence at zero wave number \((σ_1^+(0) = [-1 + √1 + 4α]/2)\), a local maximum exists and equals \(\frac{1}{2}[-1+\sqrt{1+4α}]\), and the root is purely real when \(α \geq -1/4\). However, it is complex when \(α < -1/4\), and \(-1/2\) is the value of its maximum. Also, there is a local maximum of \(σ_1^+\) and can be computed from (16). This maximum equals \(-(1 + 2\lambda k_1^2)/(1 + \lambda)\) (always less than \(-1/2\), where the root is purely real, and \(k = k_1^+\) is the solution of (16). Now we can say that, the maximum values of \(σ_1^+\) always occurs at \(k = 0\), and in this case the maximum growth rate \(Re(σ_1^+)\) can be written as

\[
Re(σ_1^+) = \begin{cases} 
(−1 + √1 + 4α)/2 & α \geq -1/4, \quad Im(σ_1^+) = 0 \\
-1/2 & α < -1/4, \quad Im(σ_1^+) = \frac{1}{2}\sqrt{|1 + 4α|} 
\end{cases}
\]

Hence, for \(α \geq -1/4\) the maximal value is \([-1 + √1 + 4α]/2\) (stationary uniform instability exists), and when \(α < -1/4\), an oscillatory uniform bifurcation arises.

### 4 Results and Conclusion

Table 2: Instabilities of (1) when \(D_v < D_u\), where \(k_1^+\) is the most unstable wave number, and \(Re(σ_*)\) is the corresponding growth rate.\(K_1 = g_v - f_u)^2 + 4f_vg_u\), \(k_2 = D_uD_v(g_v - f_u)^2 + (D_u + D_v)^2f_vg_u\), and \(K_3 = 16D_uD_vf_vg_u + (D_u + D_v)^2(g_v - f_u)^2\)

<table>
<thead>
<tr>
<th>(k_1^2)</th>
<th>(Re(σ_*))</th>
<th>(Im(σ_*))</th>
<th>Conditions</th>
<th>Bifurcation</th>
</tr>
</thead>
</table>
| 0 | \(
\frac{1}{2}(g_v + f_u) + \sqrt{|K_1|}
\) | 0 | \(g_v - f_u > 0, \quad K_2 \geq 0\) | Stationary |
| \(k_1^p\) | \(f_u + \frac{(g_v - f_u)D_u - 2D_uD_vk_1^2}{D_u + D_v}\) | 0 | \(g_v - f_u > 0, \quad K_2 < 0, K_3 > 0\) | Stationary |
| 0 | \(
\frac{1}{2}g_v + f_u
\) | \(\frac{1}{2}\sqrt{|K_1|}\) | \(g_v - f_u > 0, \quad K_3 \leq 0\) | Oscillatory |
| 0 | \(
\frac{1}{2}(g_v + f_u) + \sqrt{|K_1|}
\) | 0 | \(g_v - f_u < 0, \quad K_1 \geq 0\) | Stationary |
| 0 | \(
\frac{1}{2}(g_v + f_u) + \sqrt{|K_1|}\) | \(\frac{1}{2}\sqrt{|K_1|}\) | \(g_v - f_u < 0, \quad K_1 < 0\) | Oscillatory |

In table 2, we give a summary of the instabilities which arise in system (1).
These comprehensive results are useful when one studies a two-component reaction-diffusion systems. We redo substitutions into (21) and (23), using (9) and (14), to obtain the results in the original parameters, the diffusion coefficients $D_u$ and $D_v$, and the reaction matrix elements $f_u$, $f_v$, $g_u$ and $g_v$. The summary table shows the most unstable wave number, the corresponding eigenvalue, and for each instability the parameter regimes are identified. If the instability is stationary periodic, the most unstable wave number is $k = k_p$, which is given by

$$k_p^2 = \frac{f_u - g_v + (D_u + D_v)(-f_v g_u/D_u D_v) \frac{1}{2}}{D_u - D_v}, \quad D_u > D_v \tag{24}$$

From the summary table we want to determine the instability type for the two special cases: $\lambda = 1$ and $\lambda = 0$. For the first case, equal diffusion coefficients, i.e. $D_u = D_v$, the conditions for stationary periodic instability are not satisfied, and this is clear when we refer to figure 1 (when $\lambda = 1$, only the instabilities that arise in the two regions $R1$ and $R4$ exist). Hence unequal diffusion coefficients is a necessary condition for a stationary periodic instability to arise (see Murray [10]). When the component $u$ diffuses much faster than the component $v$, then the diffusion ratio is very small, i.e. $\lambda \simeq 0$. In case I, $g_v - f_u > 0$, there is no oscillatory uniform bifurcation; however, a stationary periodic bifurcation exists at a sufficiently large value of wave number, and this result is obvious from figure 1 (as $\lambda \to 0$, the wave number $k_*$ takes very large values). This wave number, from (24) when $D_v \to 0$ is given by $k_p^2 \simeq (-f_v g_u/D_u)^{1/2} / \sqrt{D_v}$.

We have discussed the instabilities in a two-component reaction-diffusion system with constant diffusion coefficients. The system supports three types of instabilities: stationary periodic instability, oscillatory uniform and stationary uniform bifurcation. Oscillatory periodic instability does not arise as for a reaction-diffusion system of two components the quadratic characteristic equation is $\sigma^2 + p(k^2)\sigma + q(k^2) = 0$, where $\sigma$ is the eigenvalue and $k$ is the wave number. The coefficient $p(k^2)$ is always monotonic in the wave number ($p = \beta - \gamma k^2$, where $\beta$ and $\gamma$ are real and depend on the system parameters). Thus we can say that when the eigenvalue is complex, $Re(\sigma)$ is always monotonic in $k$; hence $Re(\sigma)$ never has a maximum at a wave number $k > 0$ which is a necessary condition for the oscillatory periodic type. Thus from our study, we concluded that in a two-component reaction diffusion system the oscillatory periodic instability never arises, while the other three kinds can exist. Also we can say that Oscillatory periodic instability can arise for system of at least three variables, as in this type $Im(\sigma) \neq 0$ and $Re(\sigma)$ must a non-monotonic function of the wave number $k$. A specific three-component model, extended Brusselator model (see [14, 12]), has been studied and complex patterns ap-
peared as a result of instability of oscillatory periodic type, and in the future, a three-component systems with general kinetics can be studied and similar instability analysis can be performed.

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References


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