Exact Self-similar Perturbational Solutions of Whitham-Broer-Kaup Equations

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Abstract

In this paper, by employing the perturbational method, we obtain a class of exact self-similar solutions of the Whitham-Broer-Kaup equations. These solutions are of polynomials-type whose forms, remarkably, coincident with that given by Yuen for the other physical models, such as the compressible Euler or Navier-Stokes equations, two-component Camassa-Holm equations and viscoelastic Burgers equations.

Keywords: Whitham-Broer-Kaup equations, Analytical Solutions, Perturbational Method, Self-similar solution

1 Introduction

In this paper, we would like to construct some exact solutions for the Whitham-Broer-Kaup (WBK) equations, which are expressed by the following form [1, 2, 3]

\[
\begin{align*}
    u_t + uu_x + H_x + \beta u_{xx} &= 0 \\
    H_t + (Hu)_x + \alpha u_{xxx} - \beta H_{xx} &= 0
\end{align*}
\]

where \( u = u(x,t) \) and \( H = H(x,t) \) are the field of horizontal velocity and the height that deviate from equilibrium position of the liquid. While \( \alpha \) and \( \beta \) are real constants that represent different dispersive power. In fact, it is of interest to investigate the WBK system since it contains many distinct physical models when setting different values to
α and β. For example, if α = 0 and β ≠ 0, we obtain the classical long-wave equations that describe a shallow water wave with diffusion [3, 4]. If α = 1 and β = 0, we obtain the modified Boussinesq equations [5] and if α = 0 and β = 0, the 1-dimensional shallow water equations on a flat bottom are derived [6]. Therefore, there has been much attention paid to the WBK equations, which is manifested by a large number of literatures [7-13]. For instance, in [7], Yan et al presented the solitary wave and periodic wave solutions for the WBK equations via the improved sine-cosine method. In [8], Xie et al obtain the traveling wave solutions via the hyperbolic function method while such type solutions were also derived by Xu via the auxiliary equation method [9]. In [10], Xu et al constructed abundant solitary-wave solutions via the extend tanh-function method. In [11], Zhang et al gave some new solutions via the Lie symmetry analysis. Important contributions to the WBK equations were also made by Li [12] and Zhang et al [13] with regard to the exact solutions.

Different from the works mentioned above, here we plan to study the exact self-similar blowup solutions of WBK equations via the perturbational method [14, 15]. Recently, such type solutions have been studied extensively and intensively (see references e.g. [14, 15, 16, 17, 18, 19, 20, 21, 22]). Particularly, it was shown by Yuen in [14] that the compressible Euler or Navier-Stokes equations in $R^{1+1}$:

$$\begin{align*}
\rho_t + \rho_x u + \rho u_x &= 0 \\
\rho(ut + uu_x) + K(\rho^\gamma)_x &= \mu u_{xx}
\end{align*}$$

admits the perturbational blowup solutions $(\rho, u)$:

$$\begin{align*}
\rho^{\gamma-1}(x, t) &= \max \left\{ \rho^{\gamma-1}(0, t) - \frac{\gamma-1}{K\gamma} \left[ \hat{b}(t) + b(t) \frac{\dot{a}(t)}{a(t)} \right] x - \frac{(\gamma-1)\xi}{2K\gamma a^{\gamma+1}(t)} x^2, \ 0 \right\} \\
u(x, t) &= \frac{\dot{a}(t)}{a(t)} x + b(t)
\end{align*}$$

where $a(t), b(t)$ and $c(t)$ satisfy the following ordinary differential equations:

$$\begin{align*}
\ddot{a}(t) &= \frac{\xi}{a^{\gamma}(t)}, \quad a(0) = a_0 > 0, \quad \dot{a}(0) = a_1, \\
\ddot{b}(t) + \frac{1+\gamma}{a(t)} \dot{a}(t) &+ \left[ \frac{2\xi}{a^{\gamma+1}(t)} + (\gamma - 1) \frac{\dot{a}^2(t)}{a^2(t)} \right] b(t) = 0, \quad b(0) = b_0, \quad \dot{b}(0) = b_1, \\
\frac{\partial}{\partial t} \rho^{\gamma-1}(0, t) + \rho^{\gamma-1}(0, t) \frac{\dot{a}(t)}{a(t)} - \frac{\gamma-1}{K\gamma} \left[ \hat{b}(t) + b(t) \frac{\dot{a}(t)}{a(t)} \right] b(t) &= 0, \quad \rho(0, 0) = \alpha
\end{align*}$$

where the quantities $K$ and $\gamma$ are given by the equation (2) and $a_0, a_1, b_0, b_1, \alpha$ are arbitrary constants. Interestingly, the above solutions can provide the mathematical explanations for the drifting phenomena of some propagation wave like Tsunamis which are generated by the earthquakes [14]. We observe that the WBK equations (1) share similarities with the 1+1-dimensional compressible Euler or Navier-Stokes model (2) if we take $\gamma = 2$ and roughly treat $H$ as $\rho$ in some senses. Therefore, it is natural to inquire as whether we can construct the perturbational blowup solutions for the WBK equations (1). In this paper, we successfully apply the perturbational method to obtain the self-similar perturbational solutions.
Theorem 1 For the Whitham-Broer-Kaup equations in $\mathbb{R}^{1+1}$, there exists a class of exact perturbational solutions:

\[
\begin{aligned}
\{ & u = \frac{2k_0}{3(k_0 t + k_1)} x - \frac{3(k_0 k_2 t + k_1 k_2 - 2k_0 k_3)}{2(k_0 t + k_1)}, \\
& H = \frac{k_0^2}{9(k_0 t + k_1)^2} x^2 + \left[ \frac{k_0 k_2}{k_0 t + k_1} + \frac{k_0^2 k_3}{(k_0 t + k_1)^2} \right] x + E^*(t) \}
\end{aligned}
\]

with

\[
\begin{aligned}
\{ & \dot{E}^*(t) + \frac{2k_0}{3(k_0 t + k_1)} E^*(t) - \frac{2\beta k_0^2}{9(k_0 t + k_1)^2} - \frac{3k_0}{2} \left( k_2 - \frac{2k_0 k_3}{k_0 t + k_1} \right) \left[ \frac{k_2}{k_0 t + k_1} + \frac{k_0 k_3}{(k_0 t + k_1)^2} \right] = 0 \\
& E^*(0) = E_0^* \}
\end{aligned}
\]

where $k_1, k_2, k_3$ and $E_0^*$ are constants.

Remark 2 It is noticed that if we take

\[
a = [3(k_0 t + k_1)]^\frac{3}{2}, \quad b = -\frac{3}{2} \left( k_2 - \frac{2k_0 k_3}{k_0 t + k_1} \right),
\]

the above theorem may be readily rewritten into

\[
\begin{aligned}
\{ & u = \frac{\dot{a}}{a} x + B^*(t) = \frac{\dot{a}}{a} x + \frac{9k_3 \dot{a}}{2a} - \frac{3k_2}{2}, \\
& H = -\frac{\ddot{a}}{2a} x^2 + \frac{3k_2 \ddot{a}}{2a} x + E^* \}
\end{aligned}
\]

with $E^*$ given by (6). Interestingly, the form of the velocity $u$ coincides with that given in [14, 15, 21], which fully shows the efficiency of the method adopted here.

2 Reduction of the Whitham-Broer-Kaup Equations

Our construction process of the polynomials-type solutions follows the method of under-determined coefficients in classical differential equations. For convenience, we give the following proposition for solving the Whitham-Broer-Kaup equations (1).

Proposition 3 For the Whitham-Broer-Kaup equations (1), there exists a class of exact analytical solutions:

\[
\begin{aligned}
& u = A(t)x + B(t), \quad H = C(t)x^2 + D(t)x + E(t) 
\end{aligned}
\]

with

\[
\begin{aligned}
\{ & \dot{A}(t) + A^2(t) + 2C(t) = 0, \quad A(0) = a_0, \\
& \dot{B}(t) + A(t)B(t) + D(t) = 0, \quad B(0) = b_0, \\
& \dot{C}(t) + 3A(t)C(t) = 0, \quad C(0) = c_0, \\
& \dot{D}(t) + 2B(t)C(t) + 2A(t)D(t) = 0, \quad D(0) = d_0, \\
& \dot{E}(t) + A(t)E(t) + B(t)D(t) - 2\beta C(t) = 0, \quad E(0) = e_0, \}
\end{aligned}
\]
where \( a_0, b_0, c_0, d_0 \) and \( e_0 \) are constants.

**Proof.** We just plug the functions
\[
u = A(t)x + B(t), \quad H = C(t)x^2 + D(t)x + E(t),
\]
into the Whitham-Broer-Kaup equations to obtain the corresponding results. Insertion into the first equation, now produces
\[
u_t + uu_x + H_x + \beta u_{xx}
\]
\[
= \frac{\partial}{\partial t} [A(t)x + B(t)] + [A(t)x + B(t)] \frac{\partial}{\partial x} [A(t)x + B(t)] + \frac{\partial}{\partial x} [C(t)x^2 + D(t)x + E(t)]
\]
\[
= \dot{A}(t)x + \dot{B}(t) + A^2(t)x + A(t)B(t) + 2C(t)x + D(t)
\]
\[
= \left[ \dot{A}(t) + A^2(t) + 2C(t) \right] x + \dot{B}(t) + A(t)B(t) + D(t)
\]
\[
= 0.
\]
So we have
\[
\begin{align*}
\dot{A}(t) + A^2(t) - 2C(t) &= 0, \\
\dot{B}(t) + A(t)B(t) - D(t) &= 0.
\end{align*}
\]
Substitution into the second equation, delivers
\[
H_t + (Hu)_x + \alpha u_{xxx} - \beta H_{xx}
\]
\[
= \frac{\partial}{\partial t} [C(t)x^2 + D(t)x + E(t)]
\]
\[
+ [A(t)x + B(t)] \frac{\partial}{\partial x} [C(t)x^2 + D(t)x + E(t)]
\]
\[
+ [C(t)x^2 + D(t)x + E(t)] \frac{\partial}{\partial x} [A(t)x + B(t)]
\]
\[
- \beta \frac{\partial^2}{\partial x^2} [C(t)x^2 + D(t)x + E(t)]
\]
\[
= \dot{C}(t)x^2 + \dot{D}(t)x + \dot{E}(t) + [A(t)x + B(t)] [2C(t)x + D(t)]
\]
\[
+ A(t) [C(t)x^2 + D(t)x + E(t)] - 2\beta C(t)
\]
\[
= \dot{C}(t)x^2 + \dot{D}(t)x + \dot{E}(t) + 2A(t)C(t)x^2 + [A(t)D(t) + 2B(t)C(t)] x + B(t)D(t)
\]
\[
+ A(t)C(t)x^2 + A(t)D(t)x + A(t)E(t) - 2\beta C(t)
\]
\[
= \left[ \dot{C}(t) + 3A(t)C(t) \right] x^2 + \left[ \dot{D}(t) + 2A(t)D(t) + 2B(t)C(t) \right] x
\]
\[
+ \dot{E}(t) + A(t)E(t) + B(t)D(t) - 2\beta C(t)
\]
\[
= 0.
\]
Therefore, we have

\[
\begin{align*}
\dot{C}(t) + 3A(t)C(t) &= 0, \quad C(0) = c_0, \\
\dot{D}(t) + 2A(t)D(t) + 2B(t)C(t) &= 0, \quad D(0) = d_0, \\
\dot{E}(t) + A(t)E(t) + B(t)D(t) - 2\beta C(t) &= 0, \quad E(0) = e_0.
\end{align*}
\]

(23)

The proof is completed.

The following task is to further solve the dynamical system (10) for obtaining Theorem 1. For this purpose, the technique employed by Yuen in [14] will be adopted.

**Proof of Theorem 1.** We observe that the functions (5) take the same form as that given in Proposition 3 for the Whitham-Broer-Kaup equations (1). According to the proposition, one just needs to prove that the corresponding conditions can be satisfied. For convenience, we take

\[
A^* = \frac{\dot{a}}{a}, \quad B^* = \frac{9k_3\dot{a}}{2a} - \frac{3k_2}{2},
\]

(24)

\[
C^* = -\frac{\ddot{a}}{2a}, \quad D^* = \frac{3(k_2\dot{a} - 3k_3\ddot{a})}{2a},
\]

(25)

so we have

\[
\dot{A}^* + A^*^2 + 2C^*
\]

(26)

\[
= \frac{d}{dt} \frac{\dot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 - 2\frac{\ddot{a}}{2a} \quad (27)
\]

\[
= \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\dot{a}}{a}\right)^2 - \frac{\ddot{a}}{a} \quad (28)
\]

\[
= 0 \quad (29)
\]

\[
\dot{C}^* + 3A^*C^*
\]

(30)

\[
= -\frac{d}{dt} \frac{\dddot{a}}{a} - 3\frac{\dddot{a}}{a} \quad (31)
\]

\[
= -\frac{\dddot{a}}{2a} - \frac{\dddot{a}}{a^2} \quad (32)
\]

\[
= \frac{(3k_0 t + 3k_1)^{\frac{3}{2}}[(3k_0 t + 3k_1)^{\frac{3}{2}}]'' + 2[(3k_0 t + 3k_1)^{\frac{3}{2}}]'[(3k_0 t + 3k_1)^{\frac{3}{2}}]''}{2a^2} \quad (33)
\]

\[
= -\frac{8k_0^3(3k_0 t + 3k_1)^{-\frac{3}{2}} + 2[2k_0(3k_0 t + 3k_1)^{-\frac{1}{2}}][-2k_0^2(3k_0 t + 3k_1)^{-\frac{1}{2}}]}{2a^2} \quad (34)
\]

\[
= 0 \quad (35)
\]
In the following, we check the rest three conditions.

\[
\dot{B}^* + A^* B^* + D^* = \frac{d}{dt} \left[ \frac{9k_3 \dot{a}}{2a} - \frac{3k_2}{2} \right] + \dot{\dot{a}} \left[ \frac{9k_3 \dot{a}}{2a} - \frac{3k_2}{2} \right] + 3(k_2 \ddot{a} - 3k_3 \dot{a}) \tag{36}
\]

\[
= \frac{9k_3 \ddot{a}}{2a} - \frac{9k_3 \dot{a}^2}{2a^2} + \frac{9k_3 \dot{a}^2}{2a^2} - \frac{3k_2 \dot{a}}{2a} + \frac{3k_2 \dot{a}}{2a} - \frac{9k_3 \dot{a}}{2a} \tag{37}
\]

\[
= 0, \tag{38}
\]

\[
\dot{D}^* + 2A^* D^* + 2B^* C^* = \frac{d}{dt} \left[ \frac{3(k_2 \dot{a} - 3k_3 \ddot{a})}{2a} \right] + 2 \dot{\dot{a}} \left[ \frac{3(k_2 \dot{a} - 3k_3 \ddot{a})}{2a} \right] - \frac{2}{2a} \left[ \frac{9k_3 \ddot{a}}{2a} - \frac{3k_2}{2} \right] \tag{39}
\]

\[
= \frac{3k_2 (\ddot{a} + \frac{\ddot{a}^2}{2a^2}) - 9k_3 \left( \frac{\dddot{a}}{2a} + \frac{\ddot{a}}{a^2} \right)}{a^2} \tag{40}
\]

\[
= 3k_2 \left[ \frac{-2k_0^2}{9(k_0 t + k_1)^2} + \frac{1}{2} \left( \frac{2k_0}{3(k_0 t + k_1)} \right)^2 \right] - 9k_3 \left[ \frac{4k_0^3}{3(k_0 t + k_1)^3} + \frac{2k_0}{3(k_0 t + k_1)} \left( \frac{k_2}{k_0 t + k_1} + \frac{k_0 k_3}{(k_0 t + k_1)^2} \right) \right] \tag{41}
\]

\[
= 0, \tag{42}
\]

\[
\dot{E^*} + \frac{2k_0}{3(k_0 t + k_1)} E^* \left( t \right) - \frac{2\beta k_0^2}{3(k_0 t + k_1)^2} \frac{3k_0}{2} \left( k_2 - \frac{2k_0 k_3}{k_0 t + k_1} \right) \left( \frac{k_2}{k_0 t + k_1} + \frac{k_0 k_3}{(k_0 t + k_1)^2} \right) = 0. \tag{43}
\]

The proof is complete. ■

By using the above results, we are ready to discuss qualitatively the properties on the blowup or global existence of the perturbational solutions (5). The standard argument runs as follows:

With the aid of \( a = [3(k_0 t + k_1)]^\frac{1}{2} \), \( \ddot{a} = 2k_0 \left[ 3(k_0 t + k_1) \right]^{-\frac{1}{2}} \) and \( \dddot{a} = -2k_0^2 \left[ 3(k_0 t + k_1) \right]^{-\frac{3}{2}} < 0 \), the first and second derivative tests can thus be employed to find the extremum of \( a(t) \), \( t \in [0, +\infty) \). After that, by comparing the extremum, we can determine whether there exists a finite time \( T \in (0, +\infty) \), such that \( a(T) = 0 \). Consequently, we obtain the corresponding blowup or global results for the analytical solutions (5).

To end the discussion, we summarize our findings by the following proposition.

**Proposition 4** (1) For \( \beta > 0 \),

(1a) If

\[
\begin{cases}
  k_1 < -1 \\
  k_2 > 0
\end{cases} \quad \begin{cases}
  -1 < k_1 < 0 \\
  k_2 \leq 0
\end{cases} \quad \begin{cases}
  -1 < k_1 < 0 \\
  k_1 + \frac{k_2}{\beta} (1 - \ln \frac{k_2}{\beta}) \leq 0
\end{cases} \quad \text{or} \quad \begin{cases}
  k_1 \geq 0 \\
  k_2 < 0
\end{cases} \tag{47}
\]
solutions (5) blow up at a finite time $T$;
(1b) If
\[
\begin{cases}
k_1 < -1 \\
k_2 \leq 0
\end{cases}, \quad \begin{cases}
-1 < k_1 < 0 \\
k_2 \geq \beta
\end{cases}, \quad \begin{cases}
-1 < k_1 < 0 \\
k_1 + \frac{k_2}{\beta} (1 - \ln \frac{k_2}{\beta}) > 0
\end{cases} \quad \text{or} \quad \begin{cases}
k_1 \geq 0 \\
k_2 \geq 0
\end{cases}
\]
solutions (5) globally exist.

(2) For $\beta = 0$,
(2a) If $k_1 > 0$, $k_2 \geq 0$ or $k_1 < 0$, $k_2 \leq 0$, solutions (5) globally exist.
(2b) If $k_1 < 0$, $k_2 > 0$ or $k_1 < 0$, $k_2 > 0$, solutions (5) blow up at $T = -\frac{k_1}{k_2}$.

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