Some Asymptotic Estimations in Distribution

Theory of Distinguishable Particles and

Applications for Error-Correcting Coding

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Abstract

In this paper the authors deduce two limit theorems of distribution theory of distinguishable random particles. We analyze possibilities of application of analytical estimations for boundary estimation of potential interference-immunity and consider the particular case of multichannel system with identical channels structure, interference parameters and duplication of transmitted information packets.

Keywords: random distribution, distinguishable particles, asymptotic estimations, limit theorems, error-correcting coding, multichannel

1 Introduction

Estimation of analytical performance boundaries for different scenarios in modern telecommunication systems is one of the most difficult and actual problems in the information theory and statistical signal processing [1, 2].

This paper is concerned with adaptation of analytical results of two probability asymptotic theorems for estimation of analytical equation for performance boundaries in the specific case of information packets duplication in multichannel wireless system with uniform channel structure and similar interference conditions.

Let the system has $N$ channels and information block is translated through each channels, that is the block is translated $n$ times and each of these blocks are
coded by error-correcting code. Let \( n \) be all errors in the \( N \) channels. For \( r \geq 0 \) define \( B_r \) as an event that there is number of bit errors, which is not equal to \( r \), in the \( N \) channels. And define \( B_{\leq r} \) as an event that at least one of \( N \) channels has a number of errors which is not greater than \( r \).

Consider error-correcting code, this code can correct at least \( r \) mistakes, the type of the mistake is replacement mistake. The simplest example of such codes are Hamming codes (e.g., [3]).

Then the event \( B_{\leq r} \) means that at least in one of the \( N \) channels all errors will be corrected, therefore the information block will be translated without mistakes.

In this paper we study asymptotic behavior of probability of events \( B_r \) and \( B_{\leq r} \) for \( n \) and \( N \) go to infinity.

This task relates to distribution of distinguishable particles task [4, 5].

The structure of the paper is:

- theorem 1 gives asymptotic estimation of probability of event \( B_r \) for \( n \) and \( N \) go to infinity;
- the main result in the paper is theorem 2, which gives an asymptotic estimation of probability of event \( B_{\leq r} \) for \( n \) and \( N \) go to infinity and this has practical application as shown previously.

### 2 Asymptotic estimation of probability of the event \( B_r \)

Consider an equiprivable scheme of distribution of \( n \) distinguishable random particles into \( N \) cells.

The event \( B_r \) is that there are no \( r \) particles in every cells. Let us consider an event \( B_{\leq r} \), this event is additional event to \( B_r \) and is that at least one of the cells has \( r \) particles.

**Theorem 1.** Let \( n, N \to \infty \), such that \( N \left( \frac{n}{N} \right)^r e^{-\frac{n}{N}} \to \beta \) and \( \frac{n}{N} \to \infty \) where \( \beta < \infty \) then

\[
P(B_r) \to f(\beta),
\]

where

\[
f(\beta) = \sum_{k=0}^{\infty} (-1)^k \frac{\beta^k}{k!(kr)!}.
\]

**Proof.** Event

\[
A = \bigcup_{i=1}^{N} A_i,
\]
where the event $A_i$ is that $i$-th cell contains $r$ particles. Therefore

\[
\mathbf{P}(B_i^r) = \sum_{i=1}^{N} \mathbf{P}(A_i) - \sum_{i \neq j} \mathbf{P}(A_i \cap A_j) + \cdots + (-1)^{k+1} \sum_{i,j, k, l, s, t \leq k} \mathbf{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) + \cdots + (-1)^n \sum_{i, j, k, l, s, t \leq n-1} \mathbf{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_n}).
\]

Notice that event $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}$ is that the cells $i_1, i_2, \ldots, i_k$ have $r$ particles, and the other $n - kr$ particles are distributed over last $N - k$ cells. The number of ways of distribution $n - kr$ particles over $N - k$ cells is $(N - k)^{n-kr}$, and the number of ways of distribution $n$ particles over $N$ cells is $N^n$. The probability of event that each cells $i_l, 1 \leq l \leq k$, has particles $j_{i_1}, j_{i_2}, \ldots, j_{i_k}$ is $\frac{1}{N^{kr}}$.

Therefore

\[
\mathbf{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = C^r_n N^{-r} \left(1 - \frac{1}{N}\right)^{n-r} - 
- C^2_n N^{-2r} \left(1 - \frac{2}{N}\right)^{n-2r} + \cdots + (-1)^{k+1} C^k_n N^{-kr} \left(1 - \frac{2}{N}\right)^{n-kr} +
\]

\[
+ (-1)^{k+1} C^k_n \frac{1}{N^{kr}} \left(1 - \frac{2}{N}\right)^{n-kr}
\]

where $N_r$ such that $0 \leq n - (N_r - 1)r$ and $n - N_r r < 0$. Let $1 < k < N$. Note, that

\[
\sum_{i,j,l,s} \mathbf{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k})
\]

\[
+ \sum_{i,j,l,s} \mathbf{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{k+1}}) + \cdots
\]

\[
+ (-1)^{n-k+1} \sum_{i,j,l,s} \mathbf{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{n-1}})
\]
is probability of event
\[ U_{i \neq i', j \neq k, 1 \leq j, l \leq k} A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}. \]

Therefore
\[ \sum_{i \neq i', j \neq k, 1 \leq j, l \leq k} P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) + \sum_{i \neq i', j \neq k, 1 \leq j, l \leq k+1} P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_{k+1}}) + \ldots + +(-1)^{n-k+1} \sum_{i \neq i', j \neq k, 1 \leq j, l \leq n-1} P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_{n-1}}) = \]
\[ = P(U_{i \neq i', j \neq k, 1 \leq j, l \leq k} A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) \leq \]
\[ \leq \sum_{i \neq i', j \neq k, 1 \leq j, l \leq n-1} P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = C_N^k C_{kr}^{k-1} (1 - \frac{k}{N})^{n-kr}, k < N_r, \]
and
\[ N C_N^r \frac{1}{N^r} \left( 1 - \frac{1}{N} \right)^{n-r} - C_N^k C_{cr}^{2r} \frac{1}{N^{2r}} \left( 1 - \frac{2}{N} \right)^{n-2r} + \ldots + \]
\[ + (-1)^k C_N^{k-1} C_n^{(k-1)r} \frac{1}{N^{(k-1)r}} \left( 1 - \frac{k-1}{N} \right)^{n-(k-1)r} - C_N^k C_{kr}^{kr} \frac{1}{N^{kr}} \left( 1 - \frac{k}{N} \right)^{n-kr} < \]
\[ < P(B) < \]
\[ < N C_N^r \frac{1}{N^r} \left( 1 - \frac{1}{N} \right)^{n-r} - C_N^k C_{cr}^{2r} \frac{1}{N^{2r}} \left( 1 - \frac{2}{N} \right)^{n-2r} + \ldots + \]
\[ + (-1)^k C_N^{k-1} C_n^{(k-1)r} \frac{1}{N^{(k-1)r}} \left( 1 - \frac{k-1}{N} \right)^{n-(k-1)r} + C_N^k C_{kr}^{kr} \frac{1}{N^{kr}} \left( 1 - \frac{k}{N} \right)^{n-kr}. \]

(3)

Following the Taylor formula \( \ln(1 - x) = -x + O(x^2), \) where \( \frac{1}{x^2} O(x^2) \to \frac{1}{2} \) for \( x \to 0. \) So we have
\[ C_N^k C_{cr}^{rl} \frac{1}{N^{rl}} \left( 1 - \frac{l}{N} \right)^{n-rl} = \]
\[ = \frac{1}{l!} \left( 1 - \frac{1}{N} \right) \ldots \left( 1 - \frac{l-1}{N} \right) N^l \frac{1}{(rl)!} \left( 1 - \frac{1}{n} \right) \ldots \left( 1 - \frac{r-1}{n} \right) \left( \frac{n}{N} \right)^{rl} e^{(n-r)l \ln(1 - \frac{l}{N})} = \]
\[ = \frac{1}{l!} \left( 1 - \frac{1}{N} \right) \ldots \left( 1 - \frac{l-1}{N} \right) N^l \frac{1}{(rl)!} \left( 1 - \frac{1}{n} \right) \ldots \left( 1 - \frac{r-1}{n} \right) \left( \frac{n}{N} \right)^{rl} \times \]
\[ \times e^{(n-r)l} \left( -\frac{l}{N} O\left( \left( \frac{1}{N} \right)^2 \right) \right) \]

(4)
Some asymptotic estimations in distribution

Since $N \left( \frac{n}{N} \right)^r e^{-\frac{n}{N}} \to \beta$ then $N \left( \frac{n}{N} \right)^r e^{-\frac{n}{N}} = \beta + o(1)$ or $\ln(N) + r\ln \left( \frac{n}{N} \right) - \frac{n}{N} = \ln(\beta + o(1)).$ Therefore $\frac{n}{N} = o((\ln(N))^\alpha)$ for any $\alpha > 1.$ Hence, because (4),

$$C_N^l C_n^l \frac{1}{N^{lr}} \left( 1 - \frac{l}{N} \right)^{n-lr} \to \frac{\beta^l}{l!(lr)!},$$

for $n, N \to \infty.$

(5)

Turning in (3) $n, N \to \infty,$ and with (5), we obtain

$$\frac{\beta}{1!r!} - \frac{\beta^2}{2!(2r)!} + \cdots + \frac{(-1)^k \beta^{k-1}}{(k-1)! ((k-1)r)!} - \frac{\beta^k}{k!(kr)!} \leq$$

$$\leq \lim_{n,N\to\infty} \inf \mathbb{P}(B_r^c) \leq \lim_{n,N\to\infty} \sup \mathbb{P}(B_r^c) \leq$$

$$\leq \frac{\beta}{1!r!} - \frac{\beta^2}{2!(2r)!} + \cdots + \frac{(-1)^k \beta^{k-1}}{(k-1)! ((k-1)r)!} + \frac{\beta^k}{k!(kr)!}$$

As $\frac{\beta^k}{k!(kr)!} \to 0,$ for $k \to \infty,$ turning in this inequation $k \to \infty,$ we get

$$\lim_{n,N\to\infty} \mathbb{P}(B_r^c) = \sum_{k=0}^{\infty} (-1)^k \frac{\beta^k}{k!(kr)!}.$$ 

This equation involves (1). The theorem has been proved.

As

$$\sum_{k=0}^{\infty} (-1)^k \frac{\beta^k}{k!} = e^{-\beta},$$

for $r=0$ from the theorem 1 we have

**Corollary.** Let $r=0,$ as $n, N \to \infty$ such that $Ne^{-\frac{n}{N}} \to \beta,$ where $\beta < \infty.$

Then

$$\mathbb{P}(B_r) \to e^{-\beta}.$$ 

### 3 Asymptotic estimation of probability of the event $B_{\leq r}$

Consider event $B_{r<}$ which is that there are more than $r$ particles in each cell. Obviously, that $B = B_{0<}.$
Theorem 2. Let \( n, N \to \infty \), such that \( N \left( \frac{n}{N} \right)^r e^{-\frac{n}{N}} \to \beta \) and \( \frac{n}{N} \to \infty \) where \( \beta < \infty \), then

\[
P(B_{r<}) \to f(\beta). \tag{6}
\]

**Proof.** Let \( B_{sr} \) be an additional event to \( B_{r<} \). The event \( B_{sr} \) is that at least one of cells has more than \( r \) particles. Then \( B_{r<} = \bigcup_{k=0}^r B_k \), where event \( B_k \) is that at least one of cells has more than \( k \) particles, i.e. additional to \( B_k \). Therefore

\[
P(B_k^c) \leq P(B_{sr}) \leq \sum_{k=0}^r P(B_k^c). \tag{7}
\]

Since \( N \left( \frac{n}{N} \right)^k e^{-\frac{n}{N}} \to 0, k < r \), then according to theorem 1

\[
\lim_{n,N \to \infty} P(B_k^c) = 0, \quad k < r, \quad \text{and} \quad \lim_{n,N \to \infty} P(B_{sr}) = \sum_{k=0}^\infty \frac{(-1)^k \beta^k}{k!(kr)!}. \tag{8}
\]

Passing to a limit in (7) as \( n, N \to \infty \), with (8) we obtain

\[
\sum_{k=1}^\infty (-1)^{k+1} \frac{\beta^k}{k!(kr)!} \leq \lim_{n,N \to \infty} \inf P(B_{sr}) \leq \lim_{n,N \to \infty} \sup P(B_{sr}) \leq \sum_{k=1}^\infty (-1)^{k+1} \frac{\beta^k}{k!(kr)!}.
\]

Therefore

\[
\lim_{n,N \to \infty} P(B_{sr}) = \sum_{k=1}^\infty (-1)^{k+1} \frac{\beta^k}{k!(kr)!}. \tag{9}
\]

The equation (9) involves (6). This proves the theorem.

**Remark 1.** Let us recall that distribution scheme for \( n \) distinguishable particles over equiprobable \( n \) cells (other name is polynomial distribution) is a set of nonnegative integer variables \( \eta_1, \eta_2, \ldots, \eta_N \) which have the joint probability distribution

\[
P\{\eta_1 = k_1, \eta_2 = k_2, \ldots, \eta_N = k_N\} = \frac{n!}{k_1!k_2!\ldots k_N!},
\]

where \( k_1 + k_2 + \cdots + k_N = n \). Then the event

\[B_r = \{\omega : \eta_i(\omega) \neq r\} \]

Notice that events \( \eta_1, \eta_2, \ldots, \eta_N \) are independent.
Remark 2. Let $\xi_{ni}, 1 \leq i \leq N, n, N \in \mathbf{N}$ be sets of independent nonnegative integer random variables with the same distribution in each series $p = p(n) = \mathbf{P}\{\xi_{ni} = r\}$. Let $B_r = B_r(n,N)$ be an event that all of random variables $\xi_{ni}, 1 \leq i \leq N$, that are not equal to $r$, i.e.

$$B_r = \cap_{i=1}^{N} \{\omega: \xi_{ni}(\omega) \neq r\}.$$  

Therefore

$$\mathbf{P}(B_r) = (1 - p)^N$$

and $\mathbf{P}(B_r) \to e^{-\beta}$ as $n, N \to \infty$, such that $Np \to \beta$. Theorem 1 is an analog of this statement for random variables $\eta_1, \eta_2, ..., \eta_N$. However as following from theorem 1 and corollary as $r=0$ the situation is similar to the case of independent event $A_t$, as $r>0$ situation appreciably differs from independent case.

Conclusions and Futurework

Application of limit theorems of distribution theory of distinguishable random particles for definition of boundaries of interference-immunity for particular scheme of multichannel system has considerable potential for the field of construction of new transceivers schemes for wireless multichannel systems.

In this paper the asymptotic estimation of probability of correct decoding was obtained for the case when the information blocks are (duplicated) through multichannel system with error-correcting coding (9).

The obtained estimation allows to estimate performance boundaries for equal channel parameters and interference conditions in multichannel wireless systems for the considered scenario of packet duplication.

The future work in this field is dedicated to adaptation of considered principles for construction new types of multiantennas systems [6] operated in harsh interference environments.

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