Transition to Chaos in a Generalization of the Lotka–Volterra System Using Chirikov's Resonance Overlap Method

Yu. V. Bibik*

Dorodnicyn Computing Center
Russian Academy of Sciences, Vavilov str. 40
Moscow, 119333 Russia
*Corresponding Author

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Abstract

The onset of chaotic behavior in a generalization of the Lotka–Volterra system with three additional factors—predator satiety, predator competition for prey, and seasonal changes—is investigated. The study is based on Chirikov's resonance overlap method, which allows one to analyze the conditions of occurrence of resonances and interaction between them. Resonances emerge under the influence of the perturbation term, which models the seasonal change factor. Chirikov's resonance overlap method makes it possible to determine the value of the seasonal change factor at which the original system exhibits a chaotic behavior.

Keywords: Lotka–Volterra system, seasonal changes, Chirikov's resonance overlap method, Hamiltonian, elliptic functions
1. Introduction

In this paper, we study the mathematical biology equation proposed by Lotka (1925) and Volterra [6] (1926), which describes the interaction in predator–prey systems. Later, this equation was given the name of Lotka–Volterra system. The basic model is the classical Lotka–Volterra system to which the author added three additional factors. The first two factors are the predator satiety and the competition of predators for prey [2]. They provide a more realistic description of the interaction between species in the system. These factors can be introduced in various ways. In the present paper, they are chosen so that the action–angle coordinates can be represented in terms of elliptic integrals. The third additional factor is seasonality, which provides for a more realistic account of the interaction of the two species with the external environment. In this paper, the seasonal factor is represented by an external periodic perturbation force. This force acting upon the Lotka–Volterra system with predator satiety and competition of predators for prey may cause resonances in the system. As the seasonal parameter changes, the system can exhibit chaotic behavior. An important problem is to find the seasonal parameter values at which the system exhibits chaotic behavior. There are different approaches to this problem. In this paper, Chirikov's resonance overlap criterion [3] is used as a tool for determining these values.

In this paper, the following problems are stated:
- Analyze the onset of chaos in the system under consideration using Chirikov's resonance overlap method.
- Determine the values of the seasonal parameter at which the system exhibits chaotic behavior.
- Reveal the specific features of the application of Chirikov's resonance overlap method to the original equation and find out if it is reasonable to use this method.

The investigation of the original equation is important because it is a realistic equation of mathematical biology. It takes into account the influence of seasonal factors that cause nonlinear interacting resonances in the system; as a result, the system exhibits a chaotic behavior.

The use of seasonal factor
Conventionally, the seasonality in the theory of dynamical systems and in ecology is taken into account by considering periodic variations of the parameters. If the dynamical system was integrable before introducing the seasonal factor, it may become nonintegrable after such a parameter have been introduced. Formally, the introduction of the seasonal factor makes the dynamical system with one degree of freedom a system with 3/2 degrees of freedom. The behavior of such systems was studied, in particular, in [7].
**Chirikov's dynamic chaos theory**

According to Chirikov's theory, the chaotic behavior can occur in completely deterministic systems, and it can be described in terms of statistical mechanics. In his first seminal paper (1959) Chirikov proposed a physical criterion of the occurrence of chaotic oscillations in nonlinear Hamiltonian systems. In 1971, in collaboration with Zaslavsky he published a review on the chaotic behavior in low-dimensional systems in the journal Uspekhi Fizicheskykh Nauk (Advances in Physics). In 1979, he published paper [3] containing the foundations of the dynamic chaos theory in classical Hamiltonian systems. Using this theory, Chirikov was able to solve some important physical problems.

*The state of the art in the physics of Hamiltonian chaos* is described in books [4, 7].

**Investigation techniques:**

– Determination of the main factors of the original system needed to use Chirikov's resonance overlap method.
– The use of Chirikov's resonance overlap method for determining the value of the seasonal factor at which the system exhibits a chaotic behavior.

**The paper is organized as follows.**

1. In Section 2, Chirikov's resonance overlap method is briefly described.
2. In Section 3, the main factors of the original system needed to use Chirikov's resonance overlap method are determined. More precisely,
   – the Hamiltonian of the generalization of the Lotka–Volterra system is constructed and the perturbation term is determined (Subsection 3.1);
   – action – angle variables for the unperturbed Lotka–Volterra system are found and the perturbation term is calculated by quadratures (Subsection 3.2);
   – the action–angle variables are found in terms of elliptic integrals and the dynamic variable $\zeta$, and the perturbation term are explicitly determined (Subsection 3.3);
   – the perturbation term is expanded in the Fourier series with respect to time and angle variable (Subsection 3.4);
   – the resonance value of the action variable is found (Subsection 3.5).
3. In Section 4, the critical amplitude of the seasonal perturbation term is determined.
4. Section 5 contains conclusions.

Let us now briefly describe Chirikov's resonance overlap method.

**2. Brief Exposition of Chirikov's Resonance Overlap Method**

Chirikov's resonance overlap method [3] provides a tool for the theoretical analysis of nonlinear oscillations of Hamiltonian systems by investigating the interaction between nonlinear resonances. Chirikov proposed a parameter charac-
terizing the interaction between resonances and a criterion for determining the values of this parameters at which the system begins to exhibit a chaotic behavior. To clarify the essence of the present study, we give a derivation of Chirikov’s criterion for a system with 3/2 degrees of freedom following Zaslavsky [7]. The Hamiltonian of this system is

$$H = H_0(I) + \varepsilon V(I, \theta, t).$$  

(2.1)

Here $H_0$ is the Hamiltonian of the unperturbed system, $V(I, \theta, t)$ is the perturbation with the amplitude $\varepsilon$, $I$ is the action variable, and $\theta$ is the angle variable. The function $V$ can be expanded in the double Fourier series with respect to $\theta$ and $t$:

$$V(I, \theta, t) = \frac{1}{2} \sum_{k,j} V_{k,j}(I) e^{i(k\theta - j\varepsilon t)}, \quad V_{k,j}^* = V_{-k,-j}.  \tag{2.2}$$

With regard to this expansion, the Hamilton equations take the form

$$\dot{I} = -\varepsilon \frac{\partial V}{\partial \theta} = -\frac{i\varepsilon}{2} \sum_{k,j} kV_{k,j}(I) e^{i(k\theta - j\varepsilon t)},$$

$$\dot{\theta} = \frac{\partial H}{\partial I} = \frac{\partial H_0}{\partial I} + \varepsilon \frac{\partial V(I, \theta, t)}{\partial I} = \omega(I) + \frac{1}{2} \varepsilon \sum_{k,j} \frac{\partial V_{k,j}(I)}{\partial I} e^{i(k\theta - j\varepsilon t)},  \tag{2.3}$$

where the unperturbed oscillation frequency is

$$\omega(I) = \frac{dH_0}{dI}.  \tag{2.4}$$

The resonance conditions imply the equation

$$k\omega(I) - l\nu = 0,  \tag{2.5}$$

where $k$ and $l$ are integers and $\nu$ is the external force frequency. Therefore, we should find a pair of integers $k_0, l_0$ and the corresponding $I_0$ that solve Eq. (2.5):

$$k_0\omega(I_0) - l_0\nu = 0.  \tag{2.6}$$

Let us analyze the simplified case of isolated resonance (2.6) and neglect all the other possible resonances. In this case, we should leave in equations of motion
(2.3) only the terms with $k = \pm k_0$ and $l = \pm l_0$ satisfying the resonance condition for $I = I_0$

$$
\dot{I} = \varepsilon k_0 V_0 \sin(k_0 \theta - l_0 \varphi + \phi),
$$
\hfill (2.7)

$$
\dot{\theta} = \omega(I) + \varepsilon \frac{\partial V_0}{\partial I} \cos(k_0 \theta - l_0 \varphi + \phi),
$$
\hfill (2.8)

where $V_{k,l_0}$ is replaced with $V_0$ according to the rule

$$
V_{k,l_0} = \left| V_{k,l_0} \right| e^{i\phi} = V_0 e^{i\phi}.
$$
\hfill (2.9)

In addition, assume that the difference

$$
\Delta I = I - I_0
$$
\hfill (2.10)

is small.

To simplify system of equations (2.7), (2.8), we make the typical approximation steps:

1. Set $V_0(I) = V_0(I_0)$ on the right-hand side of system (2.7), (2.8).
2. Expand the frequency $\omega(I)$ using (2.10) as

$$
\omega(I) = \omega_0 + \omega' \Delta I,
$$
\hfill (2.11)

where

$$
\omega_0 = \omega(I_0), \quad \omega' = \frac{d\omega(I_0)}{dI}.
$$
\hfill (2.12)

3. Neglect the second-order term in $\varepsilon$ in Eq. (2.8) for the angle. As a result, system (2.7), (2.8) is reduced to the simplified form

$$
\frac{d}{dt} (\Delta I) = -\varepsilon k_0 V_0 \sin \psi,
$$
\hfill (2.13)

$$
\frac{d}{dt} \psi = k_0 \omega' \Delta I,
$$
\hfill (2.14)
The new phase
\[ \psi = k_0 \theta - l_0 m + \phi - \pi \]  
(2.15)
is introduced under the assumption that \( \omega' > 0 \). If \( \omega' < 0 \), a phase shift in the definition of \( \psi \) should be made. Equations (2.13), (2.14) can be written in Hamiltonian form
\[ \frac{d}{dt} (\Delta I) = -\frac{\delta \overline{H}}{\delta \psi}, \quad \frac{d}{dt} \psi = \frac{\delta \overline{H}}{\delta (\Delta I)}, \]  
(2.16)
where the Hamiltonian is
\[ \overline{H} = \frac{1}{2} k_0 \omega' (\Delta I)^2 - \varepsilon k_0 V_0 \cos \psi. \]  
(2.17)
The form of resonance Hamiltonian (2.17) coincides with the Hamiltonian of the nonlinear pendulum; therefore, the formulas for the nonlinear pendulum can be automatically applied to this Hamiltonian. The Hamilton equations imply the equation for the generalized coordinate \( \psi \):
\[ \frac{d^2 \psi}{dt^2} + \Omega_0^2 \sin \psi = 0. \]  
(2.18)
It gives the oscillation frequency of the generalized coordinate:
\[ \Omega_0 = (\varepsilon k_0 V_0 |\omega|)^{1/2}. \]  
(2.19)
For the nonlinear pendulum, the frequency of oscillations of the generalized coordinate \( \Omega_0 \) coincides with the frequency width of the resonance \( \Delta \omega \). The separatrix of the nonlinear pendulum is determined by the equation
\[ I_{ss} = I_r \pm (\Delta I_r) \cos \left( \frac{k \theta - l \psi + \phi}{2} \right). \]  
(2.20)
Here, the amplitude of the phase of the pendulum oscillations is given by the rule
\[ \Delta I_r = \left( \frac{\varepsilon k_0^2 V_0}{|\omega|} \right)^{1/2}. \]  
(2.21)
Since all the formulas above were derived under the assumption that $\Delta I$ is small, the frequency width of the resonance $\Delta \omega$ for small $\Delta I$ can be found by the formula

$$\Delta \omega = \omega \Delta I. \quad (2.22)$$

Now, we turn our attention to other resonances. Denote by $\delta \omega$ the distance to the adjacent resonance. To take into account the interaction between these resonances, we define the parameter $K_{ch} = \left| \frac{\Delta \omega}{\delta \omega} \right|$. It was first introduced by Chirikov to determine the degree of interaction between resonances. When resonances overlap, the phase trajectory can freely wander between them, which is an indication of the chaotic behavior of the system. This phenomenon is associated with Chirikov’s chaos criterion $K_{ch} \geq 1$. For such values of $K_{ch}$, the motion of the system becomes chaotic. Thus, the distance between the adjacent resonances $\delta \omega$ and the frequency width of the resonance $\Delta \omega$ determine the conditions under which the system starts to exhibit chaotic behavior.

**Specific features of the application of Chirikov's method in the problem under study.** In the problem studied in this paper, there is only one external force frequency $\nu$. For this reason, the constant $l_0 = 1$ in (2.6). Therefore, the adjacent resonances differ by one in the value of the index $k_0$ in (2.6). Consequently, we can use (2.6) to derive the distance between resonances $\delta \omega$ in the form

$$(k_0 - 1)(\omega_0 + \delta \omega) = \nu. \quad (2.23)$$

This formula implies that

$$\delta \omega = \frac{\omega_0}{k_0 - 1} = \frac{\nu}{k_0(k_0 - 1)}. \quad (2.24)$$

Now, the critical value of Chirikov's parameter, which will be later used to investigate the given system in Section 4, is

$$K_{ch} = \left( \frac{\Delta I}{\delta \omega} \right) = (\frac{\Delta \omega}{\delta \omega}) = \left( \frac{k_0 - 1}{\omega_0} \right) \sqrt{\nu \varepsilon}. \quad (2.25)$$

Upon the brief presentation of Chirikov’s resonance overlap method has been completed, we proceed to the definition of the main factors characterizing the dynamics of the original system.
3. Definition of the Main Factors of the Original System Needed to Apply Chirikov's Resonance Overlap Method

First, we note that the original system must satisfy the following conditions needed for Chirikov's resonance overlap method:
- the system can be formulated in terms of Hamilton equations (Hamiltonian formalism);
- action–angle variables must be defined for it;
- the perturbation term must be expandable in the Fourier series with respect to time and the angle variable in a manageable fashion.

The classical Lotka–Volterra system does not satisfy the last condition. For that reason, I propose and investigate an appropriate model consisting of an unperturbed and perturbed parts.

The unperturbed part is a generalization of the Lotka–Volterra system with regard to the predator satiety and the competition of predators for prey.

For the perturbed part, the perturbation term must be defined.

We are coming now to the specific definition of the system factors.

3.1 Construction of the Hamiltonian of the generalization of the Lotka–Volterra system and definition of the perturbation term

Since the application of Chirikov's method is based on the knowledge of the Hamiltonian of the unperturbed system and the perturbation term, we proceed to their definition. The unperturbed generalization of the Lotka–Volterra system is chosen to be integrable in terms of elliptic functions (this point will be discussed in Subsection 3.3 in detail). It is written as

\[ \dot{x} = x - \frac{xy}{(1 + \frac{1}{3}(x + y))}, \]  
\[ (3.1.1) \]

\[ \dot{y} = -y + \frac{xy}{(1 + \frac{1}{3}(x + y))}. \]  
\[ (3.1.2) \]

Here, \( x \) is the prey population, \( y \) is the predator population, and the term \( \frac{xy}{(1 + \frac{1}{3}(x + y))} \) takes into account the interaction between the species; the denominator accounts for the predator satiety and the competition of predators for prey.

The generalization of the Lotka–Volterra system with perturbation is
where $V_1$ and $V_2$ are the terms accounting for seasonality and $\varepsilon$ is the seasonality amplitude.

The perturbation terms $V_1$ and $V_2$ determine the influence of season on the species dynamics. Let us describe $V_1$ and $V_2$ in more detail. Recall that mathematically the influence of seasonality manifests itself in the dependence of the coefficients of the linear (in $x$ and $y$) terms in the generalization of the Lotka–Volterra system (3.1.1), (3.1.2).

Then, the Lotka–Volterra system of equations with account for seasonality takes the form

$$
\dot{x} = (1 + \varepsilon \cos \nu t)x - \frac{xy}{1 + \frac{1}{3}(x + y)}, \quad (3.1.5)
$$

$$
\dot{y} = -(1 - \varepsilon \cos \nu t)y + \frac{xy}{1 + \frac{1}{3}(x + y)}, \quad (3.1.6)
$$

Hence

$$
V_1 = \varepsilon x \cos \nu t, \quad (3.1.7)
$$

$$
V_2 = \varepsilon y \cos \nu t, \quad (3.1.8)
$$

where $\nu$ is the frequency of the external force.

To be able to apply Chirikov's resonance overlap method to the original system, we rewrite the unperturbed and perturbed parts in Hamiltonian form. First, we rewrite in Hamiltonian form unperturbed system (3.1.1), (3.1.2). Making the change of variables $q = \ln x$, $p = \ln y$, we obtain
\[
\dot{q} = 1 - \frac{e^p}{(1 + \frac{1}{3}(e^q + e^p))}, \quad (3.1.9)
\]

\[
\dot{p} = -1 + \frac{e^q}{(1 + \frac{1}{3}(e^q + e^p))}, \quad (3.1.10)
\]

where \( q \) is the generalized coordinate and \( p \) is the generalized momentum.

Thus, the Hamiltonian of the unperturbed generalization of the Lotka–Volterra system is

\[
H = q + p - 3 \ln \left(1 + \frac{1}{3}(e^q + e^p)\right). \quad (3.1.11)
\]

Upon the construction of the Hamiltonian of the unperturbed generalization of the Lotka–Volterra system, we define the perturbation term.

In terms of the variables \( q, p \), the perturbed system has the form

\[
\dot{q} = (1 + \varepsilon \cos \nu t) - \frac{e^q}{(1 + \frac{1}{3}(e^q + e^p))}, \quad (3.1.12)
\]

\[
\dot{p} = (1 - \varepsilon \cos \nu t) + \frac{e^q}{(1 + \frac{1}{3}(e^q + e^p))}. \quad (3.1.13)
\]

The Hamiltonian of this system is

\[
H = q + p + (p - q)\varepsilon \cos \nu t - 3 \ln \left(1 + \frac{1}{3}(e^q + e^p)\right). \quad (3.1.14)
\]

It is Hamiltonian (3.1.11) perturbed by the term

\[
\varepsilon \cos \nu t = \varepsilon (p - q) \cos \nu t. \quad (3.1.15)
\]

In this form, the perturbation term is inconvenient for the analysis. The expression \( (p - q) \) can be transformed. To this end, subtract (3.1.10) from (3.1.9). This yields
Lotka–Volterra system using Chirikov's resonance overlap method

\[
\frac{\partial}{\partial t} (q - p) = 2 - \frac{e^p}{(1 + \frac{1}{3}(e^q + e^p))} - \frac{e^q}{(1 + \frac{1}{3}(e^q + e^p))} = 2 - \frac{e^q + e^p}{(1 + \frac{1}{3}(e^q + e^p))}.
\]

\((3.1.16)\)

Make the change of variables

\[\zeta = x + y, \quad \eta = x - y.\]

\((3.1.17)\)

\((3.1.18)\)

As a result, we obtain the general form of the perturbation term

\[\bar{V} = p - q = C - 2t + \int \frac{\zeta}{1 + \frac{1}{3}\zeta} dt. \]

\((3.1.19)\)

The dynamic variable \(\zeta\) is calculated by quadratures \((3.2.17), (3.2.18)\) in Subsection 3.2 and in explicit form \((3.3.25)\) in Subsection 3.3.

The perturbation term is expanded in the Fourier series with respect to time and the angle variable in Subsection 3.4.

### 3.2. Construction of action–angle variables for the unperturbed generalization of the Lotka–Volterra system and calculation of the perturbation term by quadratures

Action–angle variables are the most convenient ones for Chirikov's resonance overlap method \([1, 5]\). For that reason, we introduce such variables for unperturbed generalization of Lotka–Volterra system \((3.1.1), (3.1.2)\).

The introduction of these variables will enable us to find the frequency of oscillations of the unperturbed system and calculate the perturbation term by quadratures.

The first step in the construction of the action–angle variables is the derivation of a quadrature expression for the variable \(\zeta\).

**Construction of the variable \(\zeta\) by quadratures.** Let us use change of variables \((3.1.17), (3.1.18)\). It enables us to transform pair of equations \((3.1.1), (3.1.2)\) to a new pair of equations.

To obtain the first equation of the new pair, we add Eqs. \((3.1.1)\) and \((3.1.2)\). The second equation of the new pair is obtained by subtracting Eq. \((3.1.2)\) from Eq. \((3.1.1)\). The result is
\[ \zeta_i = \eta_i, \quad \text{(3.2.1)} \]
\[ \eta_i = \zeta_i - \frac{2f}{1 + \frac{1}{3} \zeta_i}. \quad \text{(3.2.2)} \]

Here
\[ f = xy = \frac{1}{4} (\zeta_i^2 - \eta_i^2). \quad \text{(3.2.3)} \]

The form of the function \( f \) suggests a new transformation of the equations. Multiply Eq. (3.2.1) by \( \zeta_i \) and Eq. (3.2.2) by \( \eta_i \) to obtain
\[ \zeta_i \zeta_i = \frac{1}{2} \zeta_i^2 = \zeta_i \eta_i. \quad \text{(3.2.4)} \]
\[ \eta_i \eta_i = \frac{1}{2} \eta_i^2 = \zeta_i \eta_i - \frac{2f \eta_i}{1 + \frac{1}{3} \zeta_i}. \quad \text{(3.2.5)} \]

The right-hand sides of these equations have the common term \( \zeta_i \eta_i \). By subtracting Eq. (3.2.4) from Eq. (3.2.5), we eliminate the term \( \zeta_i \eta_i \), and the difference of \( \zeta_i^2 \) and \( \eta_i^2 \) gives \( f_i \). As a result, we obtain
\[ \frac{1}{2} \eta_i^2 - \frac{1}{2} \zeta_i^2 = -2f_i = -\frac{2f \eta_i}{1 + \frac{1}{3} \zeta_i}. \quad \text{(3.2.6)} \]

This equation has only one term on the right-hand side. We try to simplify it by dividing the left-hand and right-hand sides of (3.2.6) by \( f \).

Let
\[ p = \int_{f_i}^{f} \frac{df}{f} = \ln f. \quad \text{(3.2.7)} \]

Then, upon the division by \( f \), (3.2.6) takes the form
The right-hand side of (3.2.8) differs from the right-hand side of (3.2.1) only by the factor $1 \frac{1}{1 + \frac{1}{3} \zeta}$. By multiplying the left-hand and right-hand sides of Eq. (3.2.1) by $1 \frac{1}{1 + \frac{1}{3} \zeta}$, we obtain

$$\frac{\zeta}{1 + \frac{1}{3} \zeta} = \frac{\eta}{1 + \frac{1}{3} \zeta}. \tag{3.2.9}$$

The right-hand sides of Eqs. (3.2.9) and (3.2.8) are identical. The left-hand side of Eq. (3.2.9) can be integrated. To this end define the variable $z(\zeta)$ by

$$z = \int \frac{d\zeta}{1 + \frac{1}{3} \zeta} = 3 \ln(1 + \frac{1}{3} \zeta). \tag{3.2.10}$$

Equation (3.2.9) takes the form

$$z_t = \frac{\eta}{1 + \frac{1}{3} \zeta}. \tag{3.2.11}$$

The right-hand sides of Eqs. (3.2.8) and (3.2.11) are identical. Therefore, the right-hand side of the difference of these equations is zero. Subtract Eq. (3.2.11) from Eq. (3.2.8) to obtain

$$p_t - z_t = (p - z)_t = 0. \tag{3.2.12}$$

Thus, we have obtained the conservation law

$$\ln f - 3 \ln(1 + \frac{1}{3} \zeta) = Const = H. \tag{3.2.13}$$

Using it, we can reduce the pair of first-order equations to a single first-order equation. For this purpose, we use the fact that
\[ f = \frac{1}{4}(\zeta^2 - \eta^2). \quad (3.2.14) \]

Then, we obtain

\[ \frac{1}{4}(\zeta^2 - \eta^2) = e'' \quad \text{for} \quad \eta > 0 \]  
\[ \frac{1}{4}(\zeta^2 - \eta^2) = \frac{1}{(1 + \frac{1}{3} \zeta)^3} e'' \quad \text{for} \quad \eta < 0 \]

Now, replace \( \eta \) in this formula with the derivative \( \zeta \), using Eq. (3.2.1) for this purpose. As a result, we obtain

\[ \zeta^2 = \zeta^2 - 4e'' (1 + \frac{1}{3} \zeta)^3. \quad (3.2.16) \]

Equation (3.2.16) can be solved by separation of variables. The resulting equation is

\[ t = \int_{\zeta_1}^{\zeta} \frac{d\zeta}{\sqrt{\zeta^2 - 4e'' (1 + \frac{1}{3} \zeta)^3}} \quad (3.2.17) \]

if \( \eta > 0 \) and

\[ T - t = \int_{\zeta_1}^{\zeta} \frac{d\zeta}{\sqrt{\zeta^2 - 4e'' (1 + \frac{1}{3} \zeta)^3}} \quad (3.2.18) \]

if \( \eta < 0 \), where \( T \) is the oscillation period of the unperturbed system.

Equations (3.2.17) and (3.2.18) can be used to determine the dynamic variable \( \zeta \).

It is determined by quadratures. In explicit form, it is found in Subsection 3.3 (3.3.25).

Next, we find the value of the dynamic variable \( \eta \).

Given \( \zeta \), it is found by formula (3.2.1). Upon finding \( \zeta \) and \( \eta \), the dynamics of the unperturbed system (3.1.1), (3.1.2) turns out to be described by quadratures.

Proceed to the construction of the action-angle variables.
To this end, define the variable $w$. It can be found from formulas (3.2.17) and (3.2.18):

$$w = \int^\varphi \frac{d\zeta}{\sqrt{\zeta^2 - 4e^\eta (1 + \frac{1}{3}\eta)^3}}$$

if $\eta > 0$ and

$$w = T - \int^\varphi \frac{d\zeta}{\sqrt{\zeta^2 - 4e^\eta (1 + \frac{1}{3}\eta)^3}}$$

if $\eta < 0$.

Now, we write Eqs. (3.1.1), (3.1.2) in terms of the canonical variables $H, w$:

$$H_c = 0,$$

$$w_c = 1 = \frac{\partial \hat{H}}{\partial H}, \quad \hat{H} = H.$$  

(3.2.21)

(3.2.22)

It is clear that these equations are very similar to the equations in terms of the action–angle variables. The next step is as follows. Find the generating function of the corresponding canonical transform $S(w, I)$. The Hamilton–Jacobi equation is

$$\frac{\partial S(w, I)}{\partial w} = \hat{H}(I) = H.$$  

(3.2.23)

Its solution is

$$S(w, I) = w\hat{H}(I) + A(I).$$  

(3.2.24)

Here $A(I)$ is the constant of integration and its derivative with respect to $I$ determines the phase shift in the definition of the angle variable; for convenience, $A(I)$ can be set to zero. The equations for the canonical transform of variables are
Here \( H \) is the old momentum, \( w \) is the old coordinate, \( \theta \) is the new angle variable, and \( I \) is the new action variable. It remains to find the relation \( H = H(I) \). Note that the period \( T \) is determined by (3.2.17) and (3.2.18) as

\[
T = 2\int_{\varepsilon_1}^{\varepsilon_2} \frac{d\zeta}{\sqrt{\zeta^2 - 4e^\theta (1 + \frac{1}{3}\zeta^3)}}. \tag{3.2.27}
\]

Therefore, we know the frequency of the unperturbed system oscillations \( \omega \) determined from the oscillation period \( T \) as

\[
\omega = \frac{2\pi}{T(H)} = \frac{dH}{dI}. \tag{3.2.28}
\]

As a result, we have obtained the differential equation for determining the relation between the Hamiltonian \( H(I) \) and the action \( I \)

\[
\frac{dH(I)}{dl} = \frac{2\pi}{T(H(I))}. \tag{3.2.29}
\]

It is solved by separation of variables

\[
\int^H T(H)\delta H = 2\pi l. \tag{3.2.30}
\]

This equation determines the action variable. The angle variable is determined by the equation \( \theta = \omega w \), where the frequency of the unperturbed system oscillations is \( \omega = \frac{dH(I)}{dl} \).

Thus, we have defined the action–angle variables for the unperturbed generalization of the system.
The expression by quadratures for $\zeta$ obtained in (3.2.17), (3.2.18) can be written in the following dorm using the expression for the canonical variable $w$:

$$w(\zeta) = t. \quad (3.2.31)$$

The canonical variable $w$ is given by (3.2.19) and (3.2.20). If we determine $\zeta$ using Eq. (3.2.31) and substitute it into (3.1.19), then the perturbation term $V$ will be found by quadratures. The dynamic variable $\zeta$ will be determined explicitly in Subsection 3.3 (formula (3.3.25)).

In Section 3.3, we will show that the function $w(\zeta)$ can be written in terms of elliptic integrals (3.3.17) up to changes of variables using elementary functions.

### 3.3. Construction of the Angle Variable in Terms of Elliptic Integrals and Explicit Determination of the Dynamic Variable $\zeta$ and the Perturbation Term

To explicitly determine the dynamic variable $\zeta$, we use expression (3.2.31) for $w(\zeta)$, where $w$ is the canonical variable proportional to the angle variable.

**To construct the angle variable, we write the function $w(\zeta)$ in terms of elliptic integrals.** Consider formula (3.2.17). The form of the integrand in (3.2.17) suggests that this integral can be expressed in terms of elliptic integrals. Let us proceed to the implementation of this idea. First, we factor the integrand. To this end, we need to know the roots of the integrand. These are the roots $\zeta_{1,2,3}$ satisfying the equation

$$\zeta_{1,2,3}^2 - 4e^H(1 + \frac{1}{3}\zeta_{1,2,3})^3 = 0, \quad (3.3.1)$$

$$\zeta_1 < \zeta_2 < \zeta_3. \quad (3.3.2)$$

Formula (3.2.17) implies that

$$t = \int_{\zeta_2}^{\zeta_3} \frac{d\zeta}{\sqrt{a(\zeta - \zeta_1)(\zeta - \zeta_2)(\zeta_3 - \zeta)}}, \quad (3.3.3)$$

where $a = \frac{4e^H}{27}$.

Make the change of variables
\[ x = \zeta - \zeta_1. \] (3.3.4)

In the new variables, (3.3.3) takes the form

\[ t = \int_{\zeta_1}^{\zeta} \frac{dx}{\sqrt{ax + (\zeta - \zeta_2)((\zeta_3 - \zeta_1) - x)}}. \] (3.3.5)

Remove the terms \((\zeta_1 - \zeta_2)\) and \((\zeta_3 - \zeta_1)\) from the radicand to obtain

\[ t = \frac{1}{a^{1/2}(\zeta_2 - \zeta_1)^{1/2}(\zeta_3 - \zeta_1)^{1/2}} \int_{\zeta_1}^{\zeta} \frac{dx}{\sqrt{x - \frac{x}{(\zeta_2 - \zeta_1) - 1(1 - \frac{x}{\zeta_3 - \zeta_1})}}}. \] (3.3.6)

In (3.3.6), we want to get rid of one of the three factors in the denominator of the integrand. For this purpose, we make the change of variables

\[ z = \sqrt{x}. \] (3.3.7)

In the new variables, (3.3.6) takes the form

\[ t = \frac{1}{a^{1/2}(\zeta_2 - \zeta_1)^{1/2}(\zeta_3 - \zeta_1)^{1/2}} \int_{\zeta_1}^{\zeta} \frac{2dz}{\sqrt{(\zeta_2 - \zeta_1) - 1(1 - \frac{z^2}{\zeta_3 - \zeta_2})}}}. \] (3.3.8)

The integral in (3.3.8) is very similar to an elliptic integral. To make them completely identical, we should get rid of the denominator in one of the radicand factors. For this purpose, we make the change of variables

\[ w = \frac{z}{\sqrt{\zeta_2 - \zeta_1}}. \] (3.3.9)
In the new variables, Eq. (3.3.8) takes the form

\[
t = \frac{1}{a^{1/2}(\xi_2 - \xi_1)^{1/2}(\xi_3 - \xi_1)^{1/2}} \int_1^\infty \frac{2\sqrt{\xi_2 - \xi_1} \, dw}{\sqrt{(w^2 - 1)(1 - \frac{\xi_2 - \xi_1}{\xi_3 - \xi_2})w^2}}.
\]  

(3.3.10)

Let

\[
k^2 = \frac{\xi_2 - \xi_1}{\xi_3 - \xi_1}.
\]  

(3.3.11)

Then, formula (3.3.10) takes the form

\[
t = \frac{2}{a^{1/2}(\xi_3 - \xi_1)^{1/2}} \int_1^\infty \frac{d\tau}{\sqrt{(w^2 - 1)(1 - k^2 w^2)}}.
\]  

(3.3.12)

This expression differs from the elliptic integral only by the integration limits. To overcome this difficulty, we introduce the new parameter \( k' \) and the new variable \( \tau \)

\[
k' \tau = \sqrt{1 - k^2 w^2}.
\]  

(3.3.13)

In addition,

\[
k'^2 + k^2 = 1.
\]  

(3.3.14)

In the new variables, (3.3.12) takes the form

\[
t = \frac{2}{a^{1/2}(\xi_3 - \xi_1)^{1/2}} \int_{\frac{1}{k' \sqrt{1 - k^2 \xi_2 - \xi_1}}}^{\infty} \frac{d\tau}{\sqrt{(1 - \tau^2)(1 - k^2 \tau^2)}}.
\]  

(3.3.15)
Now, the limits of integration are within the required range. However, the integral above is the difference of two elliptic integrals. Let us rewrite (3.3.15) as

$$t = \frac{2}{a^{1/2}(\xi_3 - \xi_1)^{1/2}} \int_0^1 \frac{d\tau}{\sqrt{(1 - \tau^2)(1 - k^2 \tau^2)}} - \frac{1}{k} \frac{1}{\sqrt{\frac{\xi_3 - \xi_1}{\xi_2 - \xi_1}}} \int_0^1 \frac{d\tau}{\sqrt{(1 - \tau^2)(1 - k^2 \tau^2)}}. \quad (3.3.16)$$

Formula (3.3.16) can be rewritten as

$$t = \frac{2}{a^{1/2}(\xi_3 - \xi_1)^{1/2}} K\left(\frac{\pi}{2}, k'\right) - \frac{2}{a^{1/2}(\xi_3 - \xi_1)^{1/2}} \int_0^1 \frac{d\tau}{\sqrt{(1 - \tau^2)(1 - k^2 \tau^2)}}. \quad (3.3.17)$$

Here, $K\left(\frac{\pi}{2}, k'\right)$ is the complete elliptic integral of the first kind. Thus, the function $w(\zeta)$ is written in terms of elliptic integrals. Up to a constant, it coincides with the angle variable. Thus, we have completed the construction of the angle variable. **Next, we determine the dynamic variable $\zeta$ in explicit form.**

From (3.3.17), we obtain the following expression for the oscillation period of unperturbed system (3.1.1), (3.1.2):

$$T = \frac{4}{a^{1/2}(\zeta_3 - \zeta_1)^{1/2}} K\left(\frac{\pi}{2}, k'\right). \quad (3.3.18)$$

Hence, the oscillation frequency of the unperturbed system is $\omega = \frac{2\pi}{T}$. Note that we have determined the frequency as a function of the Hamiltonian appearing in the roots $\zeta_3, \zeta_2, \zeta_1$ (3.3.1) rather than as a function of the action $I$; however, this is quite sufficient for the purposes of the further study. Recall that $H$ and $I$ are
related by quadrature relation (3.2.30). We now derive an expression for the
dynamic variable $\zeta$ from formula (3.3.17). To this end, we rewrite (3.3.17) as

$$
\frac{d^{1/2}(\xi_3 - \xi_1)^{1/2}}{2} t = K\left(\frac{\pi}{2}, k'\right) - \frac{1}{k^2} \int_{t_1}^{t_2} \frac{d\tau}{\sqrt{(1 - \tau^2)(1 - k^2 \tau^2)}}.
$$

(3.3.19)

By inverting the integral on the right-hand side of (3.3.19), we obtain

$$
\frac{1}{k^2} \sqrt{1 - k^2 \frac{\xi - \xi_1}{\xi_2 - \xi_1}} = \text{sn}(K\left(\frac{\pi}{2}, k'\right) - \frac{d^{1/2}(\xi_3 - \xi_1)^{1/2}}{2} t).
$$

(3.3.20)

Now, the elementary function of $\zeta$ is represented in terms of the Jacobi elliptic
function. Let us derive an expression for $\zeta$ from (3.3.20). Square both sides of
(3.3.20) and multiply them by $k^2$ to obtain

$$
1 - k^2 \frac{\xi - \xi_1}{\xi_2 - \xi_1} = k^2 \text{sn}^2\left(K\left(\frac{\pi}{2}, k'\right) - \frac{d^{1/2}(\xi_3 - \xi_1)^{1/2}}{2} t\right).
$$

(3.3.21)

The left-hand side of (3.3.21) can be written as

$$
1 - k^2 \frac{\xi - \xi_1}{\xi_2 - \xi_1} = 1 - \frac{\xi_2 - \xi_1}{\xi_3 - \xi_1} \frac{\xi - \xi_1}{\xi_3 - \xi_1} = 1 - \frac{\xi - \xi_1}{\xi_3 - \xi_1} = \frac{\xi_3 - \xi}{\xi_3 - \xi_1}.
$$

(3.3.22)

Furthermore, take into account the fact that

$$
k'^2 = 1 - k^2 = 1 - \frac{\xi - \xi_1}{\xi_2 - \xi_1} = 1 - \frac{\xi_2 - \xi_1}{\xi_3 - \xi_1} = \frac{\xi_3 - \xi}{\xi_3 - \xi_1}.
$$

(3.3.23)
Then

\[
\frac{\zeta_3 - \zeta_2}{\zeta_3 - \zeta_1} = \frac{\zeta_3 - \zeta_1}{\zeta_3 - \zeta_1} \cdot sn^2(K(k', k'') - \frac{d^{1/2}(\zeta_3 - \zeta_1)^{1/2}}{2}) t. \tag{3.3.24}
\]

Formula (3.3.24) can be rewritten as

\[
\zeta = \zeta_3 - (\zeta_3 - \zeta_2) \cdot sn^2(K(k', k'') - \frac{d^{1/2}(\zeta_3 - \zeta_1)^{1/2}}{2}) t. \tag{3.3.25}
\]

Thus, we have obtained the dynamic variable \( \zeta \) in explicit form (3.3.25). Using the explicit expression for \( \zeta \), we can also explicitly determine the perturbation term \( \nabla \).

### 3.4. Expansion of the perturbation term in the Fourier series with respect to time and the angle variable

To expand the perturbation term in the Fourier series with respect to time and the angle variable, we find the Fourier coefficients \( C_n \) for the perturbation term. Find the Fourier coefficients of the function \( sn^2(t) \) in (3.3.25). Since \( sn^2(t) \) is an even function, its Fourier expansion has the form

\[
sn^2(K(k) x) = \sum_{n=0}^{\infty} C_n \cos(nx), \tag{3.4.1}
\]

\[
C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} sn^2(K(k) x) dx. \tag{3.4.2}
\]

The coefficient \( C_0 \) will not be used below; so we proceed to the calculation of \( C_n \) for \( n = 1, \infty \).

\[
C_n = \frac{1}{\pi} \int_{-\pi}^{\pi} sn^2(K(k) x) \cos nx dx, \quad n = 1, \infty \tag{3.4.3}
\]

Or

\[
C_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \tag{3.4.4}
\]
where

\[ f(x) = \text{sn}^2 \left( \frac{K(k)}{\pi} x \right) \exp(\text{in} x). \tag{3.4.5} \]

To calculate the integral on the right-hand side of (3.4.4), we use complex analysis. Move the integration from the segment \([-\pi, \pi]\) on the real axis to the complex plane on the contour \(L\). Consider the integral over the closed contour \(L\)

\[ I = \oint_L f(z)\,dz. \tag{3.4.6} \]

To find coefficients (3.4.4), consider the integral over the closed contour in which \(L\) is the parallelogram with the vertices at the points of the complex plane \(-\pi, \pi, \pi + \pi\tau, -\pi + \pi\tau\), where

\[ \tau = \frac{i2K(k')}{K(k)}. \tag{3.4.7} \]

To calculate the integral, we should find a relation between integrals (3.4.6) and (3.4.4). To this end, consider the integrals over the edges of the parallelogram separately. Since the function \(f(z) = \text{sn}^2 \left( \frac{K(k)}{\pi} z \right) \exp(iz)\) is \(2\pi\)-periodic, we obviously have

\[ \int_{-\pi}^{\pi} f(z)\,dz + \int_{-\pi+\pi\tau}^{\pi+\pi\tau} f(z)\,dz = \int_{-\pi}^{\pi} f(z-2\pi)\,dz + \int_{-\pi+\pi\tau}^{\pi+\pi\tau} f(z)\,dz = \int_{-\pi}^{\pi} f(v)\,dv + \int_{-\pi+\pi\tau}^{\pi+\pi\tau} f(z)\,dz \equiv 0. \tag{3.4.8} \]

Therefore, integral (3.4.6) can be written as

\[ I = \int_{-\pi}^{\pi} f(z)\,dz + \int_{\pi+\pi\tau}^{-\pi+\pi\tau} f(z)\,dz. \tag{3.4.9} \]

Let us change the variable \(z\) in the second integral in (3.4.9) for \(z + \pi\tau\). Then we can use the periodicity of \(\text{sn}(z)\). It implies
In addition, we have for the exponential function the relation

\[ \exp(in(z + \pi \tau)) = q^{2n} \exp(inz), \]  

where

\[ q = \exp(i\pi \frac{\tau}{2}). \]  

Using (3.4.10) and (3.4.11), we obtain from (3.4.9)

\[ I = (1 - q^{2n}) \int_{-\pi}^{\pi} f(z) \, dz. \]  

Thus, we have related the integral over the closed contour in the complex plane to the integral of \( f(z) \) over the segment \([-\pi, \pi]\). Hence,

\[ C_n = \frac{1}{\pi} \frac{I}{1 - q^{2n}}. \]  

The integral over a closed contour \( I \) in the complex plane can be calculated using complex analysis techniques. For this purpose, one should analyze the poles of the integrand. The function \( f(z) \) has a second-order pole at the point \( \pi \frac{\tau}{2} \). Below, we will use the fact that

\[ sn(z + iK(k')) = \frac{1}{k' sn}(z). \]

Thus, we know the location and the type of the pole of the function \( sn(z + iK(k')) \).

We have
\[ sn^2\left(\frac{K(k)}{\pi}\left(\frac{\pi}{2} + z\right)\right) \exp\left(in\left(\frac{\pi}{2} + z\right)\right) = \frac{1}{k^2 sn^2\left(\frac{K(k)}{\pi}\right)} \exp\left(in\left(\frac{\pi}{2} + z\right)\right) = \]

\[ \frac{1}{k^2 sn^2\left(\frac{K(k)}{\pi}\right) \exp\left(-in\left(\frac{\pi}{2} + z\right)\right)} = \frac{\left[ksn\left(\frac{K(k)}{\pi}\right) \exp\left(-in\frac{z}{2}\right)\right]^2}{q^n}. \]

(3.4.16)

It is seen that \( z = 0 \) is a second-order pole. To calculate the integral, we should find the residue at this pole. Find the first- and second-order derivatives of the function \( g(z) = ksn\left(\frac{K(k)}{\pi}\right) \exp\left(-in\frac{z}{2}\right) \).

The first derivative of \( g(z) \) is

\[ \frac{d}{dz} g(z) = k[cn\left(\frac{K(k)}{\pi}\right) \exp\left(in\frac{z}{2}\right) + sn\left(\frac{K(k)}{k}\right) \left(\frac{in}{2}\right) \exp\left(-in\frac{z}{2}\right)]. \]

(3.4.17)

The second-order derivative of \( g(z) \) is

\[ \frac{d^2}{dz^2} g(z) = -k sn\left(\frac{K(k)}{\pi}\right) (am'\left(\frac{K(k)}{\pi}\right) - \frac{K(k)^2}{\pi}) \exp\left(-in\frac{z}{2}\right) + \]

\[ + k cn\left(\frac{K(k)}{\pi}\right) \left(\frac{K(k)}{\pi}\right) \exp\left(-in\frac{z}{2}\right) - \]

\[ - 2k cn\left(\frac{K(k)}{\pi}\right) am\left(\frac{K(k)}{\pi}\right) \left(\frac{K(k)}{\pi}\right) \left(\frac{in}{2}\right) \exp\left(-in\frac{z}{2}\right) + \]

\[ + k sn\left(\frac{K(k)}{\pi}\right) \left(\frac{in}{2}\right)^2 \exp\left(-in\frac{z}{2}\right). \]

(3.4.18)

Now we can find the values of these derivatives at zero:

\[ \frac{d^2}{dz^2} g(z) \bigg|_{z=0} = -k \frac{K(k)}{\pi} in, \]

(3.4.19)
\[
\frac{d}{dz} g(z) \bigg|_{z=0} = k \frac{K(k)}{\pi}.
\]  

(3.4.20)

Thus, we have the following expression for \( f(z) \):

\[
f(z) = \frac{q^n}{[k \frac{K(k)}{\pi} z - k \frac{K(k)}{2\pi} i n z^2 + O(z^3)]^2}.
\]  

(3.4.21)

Rewrite this formula in the form of Laurent series

\[
f(z) = \frac{q^n}{[k \frac{K(k)}{\pi} z - k \frac{K(k)}{2\pi} i n z^2 + O(z^3)]^2} = q^n \left(1 + \frac{i n}{2} z + O(z^2)\right)^2 = q^n \pi^2 \left(\frac{1}{z} + \frac{i n}{2} + O(z)\right)^2 = q^n \pi^2 \left(\frac{1}{z^2} + \frac{i n}{z} + O(1)\right).
\]  

(3.4.22)

Now, we can find the residue of \( f(z) \) at the point \( \frac{\pi}{2} \):

\[
\text{res}_f \left(\frac{\pi}{2}\right) = \left(\frac{\pi}{k(K(k))}\right)^2 q^n i n.
\]  

(3.4.23)

Therefore, we find the Fourier coefficient \( C_n \) as

\[
C_n = \frac{1}{\pi} \frac{2\pi i \text{res}_f \left(\frac{\pi}{2}\right)}{1 - q^{-2n}} = -\left(\frac{\pi}{k(K(k))}\right)^2 \frac{2nq^n}{1 - q^{-2n}}.
\]  

(3.4.24)

Thus, we have obtained the Fourier expansion for the function \( sn^2\left(\frac{K(k)}{\pi} x\right) \).

The Fourier expansion of the function \( sn^2(K(k') - \frac{a^{1/2}(\zeta_3 - \zeta_1)^{1/2}}{2} t) \) can be found using (3.3.25):
Lotka–Volterra system using Chirikov’s resonance overlap method

\[ sn^2(K(k') - \frac{a^{1/2}(\xi_3 - \xi_1)^{1/2}}{2}) = C_0 + \sum_{n=1}^{\infty} C_n \cos(\frac{\pi n}{K(k')} a^{1/2}(\xi_3 - \xi_1)^{1/2}) t. \]

(3.4.25)

Hence, the Fourier series for the function \( \zeta \) is

\[ \zeta = \xi_3 - (\xi_3 - \xi_2)C_0 - \sum_{n=1}^{\infty} (\xi_3 - \xi_2)C_n \cos(\frac{\pi n}{K(k')} a^{1/2}(\xi_3 - \xi_1)^{1/2}) t. \]

(3.4.26)

Next, we can obtain the Fourier series for the function \( \overline{V} = C - 2t + \int_{1+\frac{1}{3}\zeta}^{t} \zeta \, dt \).

Linearize \( \overline{V} \) with respect to \( \zeta \). If the oscillation amplitude is small, \( \zeta \) fluctuates about \( \zeta = 6 \). Consider the function

\[ G = \frac{\zeta}{1 + \frac{1}{3} \zeta}. \]

(3.4.27)

Represent \( \zeta \) as \( \zeta = 6 + (\zeta - 6) \), and expand the function \( G(\zeta) \) in the Taylor series about the point \( \zeta = 6 \):

\[ G(\zeta) = G(6 + (\zeta - 6)) = G(6) + G'(6)(\zeta - 6) + .... \]

(3.4.28)

Next, we use the fact that \( G(6) = 2 \) and \( G'(6) = \frac{1}{9} \) to write the expansion of \( G(\zeta) \) in the form

\[ G(\zeta) = 2 + \frac{1}{9} (\zeta - 6) + .... \]

(3.4.29)

Take into account the fact that \( \overline{V} \) has the form

\[ \overline{V} = C - 2t + \int G(\zeta) = C - 2t + 2t + \frac{1}{9} \int (\zeta - 2 \zeta + .... \]

(3.4.30)
The expansion of $\bar{V}$ in the Taylor series in powers of $\zeta$ suggests the use of a model equation. In this paper, we use the model $\bar{V}$ defined as $\bar{V} = C - \lambda t + \frac{1}{9} \int_0^t \zeta \, dt$.

The terms that are linear in $t$ can be eliminated:

$$\bar{V} = C - \frac{1}{9} (\zeta_2 - \zeta_3) \sum_{n=1}^{\infty} (-1)^n C_n \frac{K(k')2}{\pi a^{1/2}(\zeta_3 - \zeta_1)^{1/2}} \sin \left( \frac{\pi a^{1/2} (\zeta_3 - \zeta_1)^{1/2}}{2} n \right) \frac{2 \pi}{k' K(k')} \frac{2 q^n}{1 - q^{2n}} \sin \left( \frac{\pi a^{1/2} (\zeta_3 - \zeta_1)^{1/2}}{2} \right) t.$$  

(3.4.31)

Now, we can plug (3.4.24) into (3.4.31) to obtain the ultimate expression for the perturbation term $\bar{V}$

$$\bar{V} = C + \sum_{n=1}^{\infty} \frac{1}{9} \frac{\zeta_2 - \zeta_3}{a^{1/2}(\zeta_3 - \zeta_1)^{1/2}} (-1)^n \frac{2 \pi}{k' K(k')} \frac{2 q^n}{1 - q^{2n}} \sin \left( \frac{\pi a^{1/2} (\zeta_3 - \zeta_1)^{1/2}}{2} \right) t.$$  

(3.4.32)

Thus, we have obtained the Fourier expansion of $\bar{V}$ with respect to the angle variable $\theta$. Next, we want to obtained the expansion in the double Fourier series with respect to the variables $\theta$ and $\nu$. To this end, define the new phase $\bar{\nu} = \frac{\pi}{2} + \frac{\pi a^{1/2}}{K(k')} (\zeta_3 - \zeta_1) \frac{(\zeta_3 - \zeta_1)^{1/2}}{t}$. Then the perturbation term, which includes $\cos \nu$, takes the form

$$\bar{V} \cos \nu = \sum_{n} V_{n,1} \cos (\bar{\nu} - \nu) + V_{n,-1} \cos (\bar{\nu} + \nu).$$  

(3.4.33)
Formula (3.4.33) implies the expressions for the Fourier coefficients \( V_{n,l} \) in the double expansion with respect to the time \( \tau \) and the angle variable \( \theta \):

\[
V_{n,1} = \frac{1}{9} \frac{\frac{1}{a^{1/2}}(\zeta_2 - \zeta_1)}{\frac{1}{k^2} (k')^2} (\frac{4\pi}{K(k')} \frac{(1-a^{1/2})}{(1-a^{2n})}).
\]

(3.4.34)

Formula (3.4.34) implies the expression for the magnitude of this coefficient \( V_0 \):

\[
V_0 = \frac{1}{9} \frac{\frac{1}{a^{1/2}}(\zeta_2 - \zeta_1)}{\frac{1}{k^2} (k')^2} \frac{4\pi}{K(k')} \frac{q^n}{(1-a^{2n})}.
\]

(3.4.35)

Thus, we have obtained the Fourier coefficients \( V_{n,l} \) in the expansion of perturbation term (3.4.34) with respect to time and the angle variable. The frequency \( \omega \) of the unperturbed oscillations was earlier determined in terms of the period by formula (3.3.18).

### 3.5. Determination of the resonance value of the action variable

Now we want to determine the key quantities affecting the emergence of nonlinear resonances and interaction between them in the original system. Consider in detail the determination of the resonance value of the action variable. For this purpose, return to Subsection 3.2, where the action–angle variables were introduced. Consider the following formulas:

- (3.2.27) for determining the oscillation period of the unperturbed system \( T \);
- (3.2.28) for determining the oscillation frequency of the unperturbed system \( \omega \) from the period \( T \);
- (3.2.29) and (3.2.30), which determine the relation of the Hamiltonian with the action variable.

These formulas confirm that the quantities determined by them are interrelated because the Hamiltonian \( H \) and the action variable \( I \) are related by quadrature formula (3.2.30). If we look attentively at formula (3.2.27) for determining the
period, we see that, in order to determine the action variable, it is more convenient to operate in terms of the variable $\alpha = 4e^{\mu}$, which appears in the expression for the period $T$. Let us use this fact and try to determine the resonance value $I_0$ of the action variable $I$ in terms of the variable $\alpha$. The initial formula for determining the resonance value $I_0$ of the action variable $I$ is

$$k_0 \omega(I_0) - I_0 \nu = 0.$$  \hspace{1cm} (3.5.1)

Since we are going to manipulate in terms of $\alpha$, formula (3.5.1) can be rewritten as

$$k_0 \omega(\alpha) - I_0 \nu = 0.$$  \hspace{1cm} (3.5.2)

In terms of the period $T(I_0)$, it can be written as

$$\frac{k_0}{T(\alpha)} \frac{2\pi}{T(\alpha)} - I_0 \nu = 0$$  \hspace{1cm} (3.5.3)

or

$$T_0(\alpha) = 2\pi \frac{k_0}{I_0 \nu}.$$  \hspace{1cm} (3.5.4)

To solve Eq. (3.5.2) for $\alpha$, we must take into account the following facts. The frequency is determined based on the period. The quantity $\alpha$ appears in the expression for the period through the roots $\zeta_1, \zeta_2, \zeta_3$ and the quantity $a$. The roots $\zeta_1, \zeta_2, \zeta_3$ were introduced in Subsection 3.3 in formula (3.3.1) to represent the action variable in terms of elliptic integrals. It was found out in Subsection 3.2 that the oscillation period of the unperturbed system $T$ is

$$T = \frac{4}{a^{3/2} (\zeta_3 - \zeta_1)^{3/2}} K(\frac{\pi}{2}, k'),$$  \hspace{1cm} (3.5.5)
where

\[
    k' = \frac{\xi_3 - \xi_2}{\xi_3 - \xi_1}. \quad (3.5.6)
\]

The difficulty is in obtaining an explicit representation of the roots \( \xi_1 < \xi_2 < \xi_3 \) of the cubic equation \( \xi^2 - \alpha (1 + \frac{1}{3} \xi)^3 = 0 \) (3.3.1). It is difficult to obtain an explicit expression for them depending on \( \alpha \). For that reason, we approximate the equation \( \xi^2 - \alpha (1 + \frac{1}{3} \xi)^3 = 0 \) by two quadratic equations with a similar dependence of the roots on \( \alpha \). The given cubic equation can be written in the form

\[
    \alpha = \frac{\xi^2}{(1 + \frac{1}{3} \xi)^3}. \quad (3.5.7)
\]

The idea is as follows. The roots of the cubic equation are the intersection points of line 1, which is parallel to the axis \( \xi \) at the height \( \alpha \) with plot 2 of the function \( \frac{\xi^2}{(1 + \frac{1}{3} \xi)^3} \) (see Fig. 1).

Computations for the right-hand side of Eq. (3.5.7) show that, for positive \( \xi \), this function first increases starting from zero, then attains its maximum \( 4/3 \) at \( \xi = 6 \), and then decreases to zero at infinity. For negative \( \xi \), this function begins at zero and increases with the increasing magnitude of \( \xi \) to infinity at the point \( \xi = -3 \). Such a behavior of this function implies that, if \( \alpha \) is such that the equation has three roots, then \( \alpha \) must be in the interval between 0 and \( 4/3 \). In Fig. 1, these roots correspond to the intersection points of curves 1 and 2.

Consider two quadratic equations for \( \alpha \) that model cubic equation (3.5.7) for \( \xi > 0 \) and \( \xi < 0 \).

For \( \xi > 0 \),

\[
    \alpha = \frac{\xi}{\lambda + \beta \xi + \xi^2 / 27}. \quad (3.5.8)
\]
In this equation, the asymptotic behavior of the right-hand side as $\zeta \to \infty$ is the same as for Eq. (3.5.7). It remains to determine the constants $\lambda$ and $\beta$. The model quadratic equation must have the maximum $4/3$ at $\zeta = 6$. Hence, $\lambda = 4/3$ and $\beta = 11/36$.

For $\zeta < 0$,

$$\alpha = \frac{\zeta}{(1 + \frac{1}{3} \zeta)^2}. \quad (3.5.9)$$

Let us rewrite the two model equations in a more traditional form. The first equation is

$$\frac{1}{27} \zeta^2 + \left(\frac{11}{36} - \frac{1}{\alpha}\right) \zeta + \frac{4}{3} = 0. \quad (3.5.10)$$

Its roots are

$$\zeta_1, \zeta_2 = \frac{27}{2} \left[ \left(\frac{1}{\alpha} - \frac{11}{36}\right) \pm \sqrt{\left(\frac{11}{36} - \frac{1}{\alpha}\right)^2 - \frac{16}{81}} \right]. \quad (3.5.11)$$

The second quadratic equation is

$$\frac{1}{9} \zeta^2 + \frac{2}{3} \zeta + 1 + \frac{\zeta}{\alpha} = 0. \quad (3.5.12)$$

Its root is

$$\zeta_1 = -\frac{9}{2} \left(-\frac{2}{3} + \frac{1}{\alpha}\right) + \frac{4}{3} \alpha^2 + \frac{1}{\alpha^2}. \quad (3.5.13)$$

The roots $\zeta_1, \zeta_2, \zeta_3$ appear in formula (3.5.5) for the period $T$ both directly and in the combination $k' = \frac{\zeta_3 - \zeta_2}{\zeta_3 - \zeta_1}$. Formulas (3.5.11) and (3.5.13) determine the dependence on the parameter $\alpha$. Therefore, we have obtained the dependence of the oscillation period of the unperturbed system $T$ on $\alpha$. All the data needed to solve Eq. (3.5.4) for $\alpha$ and determining the resonance value of $\alpha$ are obtained.
However, since Eq. (3.5.4) includes an elliptic integral, the resonance value of $\alpha$ was found using a computer. The resulting dependence of $\alpha$ on the frequency of the external force $\nu$ is displayed in Fig. 2. This dependence can be found for various values of $k_0$. Below, the dependence of $\alpha$ on the frequency of the external force $\nu$ is denoted by $\alpha_{k_0}(\nu)$. The plots of the functions $\alpha_1(\nu)$, $\alpha_2(\nu)$, $\alpha_3(\nu)$, $\alpha_4(\nu)$, and $\alpha_5(\nu)$ are depicted in Fig. 2 by curves 1, 2, 3, 4, and 5, respectively. Thus, we have obtained the resonance value of $\alpha$. Next, we can calculate the derivative of the frequency $\omega$ with respect to $I$ as a function of $\nu$ and the Fourier coefficient $V_{k_0,1}$ of the perturbation term. They will be found in Section 4.

4. Determination of the Critical Amplitude of the Seasonal Perturbation Term

Knowing the data found in the preceding sections, we now determine the values of the seasonal factor for which system (3.1.5), (3.1.6) exhibits a chaotic behavior. The presence of the seasonal perturbation term leads to resonances in the system. For the critical values of the seasonal factor $\varepsilon$, the resonances overlap and the system exhibits chaotic behavior. We determine the critical value of the seasonal factor $\varepsilon$ using Chirikov's parameter ($K_{ch}$) defined as

$$K_{ch} = \Delta I = \frac{\Delta \omega}{\partial \omega}.$$  \hfill (4.1)

where

$\Delta I$ is the width of the resonance, $\partial I$ is the distance between the resonances, $\Delta \omega$ is the frequency width of the resonance, and $\partial \omega$ is the frequency distance between the resonances. The frequency width of the resonance is found by the rule

$$\Delta \omega = \Omega_0 = (\beta k_0^2 V_{0,k_0} |\varphi|)^{1/2}. \hfill (4.2)$$

Here, $V_{0,k_0}$ is the magnitude of the Fourier coefficient of the perturbation term $V_{k_0,1}$.

The frequency distance between the resonances is
\[ \delta \omega = \frac{l_0 V}{k_0 (k_0 - 1)}. \]  

Substitute the quantities \( \Delta \omega \) and \( \delta \omega \) into (4.1). The resonances overlap when

\[ K_{ch} = 1. \]  

With regard to (4.4), formula (4.1) can be written as

\[ \Delta \omega = \delta \omega. \]  

The frequency width of the resonance can be replaced with the half-sum of two adjacent resonance frequencies

\[ \frac{\Delta \omega_{k_0}}{2} + \frac{\Delta \omega_{(k_0 - 1)}}{2} = \delta \omega. \]  

The amplitude of the seasonal perturbation term \( \varepsilon \) appears in formula (4.2) for the resonance frequency width. We find it from (4.6):

\[ \varepsilon = 4 \delta \omega^2 [(k_0^2 |\omega'(\alpha_{k_0})|\sqrt{k_0} + (k_0 - 1)^2 |\omega'(\alpha_{k_0 - 1})|\sqrt{k_0 - 1})^{-1}]^2. \]  

Formula (4.7) includes the derivative of the frequency \( \omega \) with respect to \( I \) rather than the frequency itself. In this paper, we use the Hamiltonian \( H \) and the variable \( \alpha \) instead of the action variable \( I \). For that reason, we change the derivative of \( \omega \) with respect to \( I \) for the derivative of \( \omega \) with respect to \( \alpha \). The factor \( \omega' \) can be transformed as

\[ \frac{d\omega}{dl} = \frac{d\omega}{dH} \frac{dH}{dl} = \frac{d\omega}{d\alpha} \frac{1}{d\alpha} \frac{d\omega}{dH} = \frac{d\omega}{d\alpha} \omega = \frac{d\omega}{d\alpha} \alpha \omega. \]
Here, \( \frac{d\omega}{d\alpha} \) has the form

\[
\frac{d\omega}{d\alpha} = \frac{d\omega}{dv} \frac{dv}{d\alpha} = \frac{l_0}{k_0} \frac{dv}{d\alpha} = \frac{l_0}{k_0} \frac{1}{\frac{d\alpha}{dv}}.
\] (4.9)

As a result, we obtain

\[
\frac{d\omega}{dl} = \frac{l_0^2}{k_0^2} \alpha(v) \frac{1}{\frac{d\alpha}{dv}}.
\] (4.10)

Here we used the dependence \( \alpha(v) \) obtained in the preceding section.

Consider \( V_{0,k_0} \), which is the absolute value of the Fourier coefficient of the perturbation term \( V_{0,1} \):

\[
V_{0,k_0} = \frac{1}{9} \frac{q^{k_0}}{a^{1/2} (\xi_3 - \xi_2)^{1/2} k^{1/2} K(k') (1 - q^{2k_0})} = \frac{1}{9} \frac{(\xi_3 - \xi_2) 16\pi}{a K(k') (1 - q^{2k_0})} =
\]

\[
= \frac{1}{9} \frac{16\pi}{aT} \frac{q^{k_0}}{(1 - q^{2k_0})} = \frac{1}{9} \frac{l_0}{a k_0} \frac{\nu}{v} \frac{q^{k_0}}{(1 - q^{2k_0})} = \frac{8}{9} \frac{l_0}{k_0} \frac{\nu}{\alpha_{k_0}} \frac{q^{k_0}}{(1 - q^{2k_0})}.
\] (4.11)

The product \( \frac{d\omega}{dl} V_{0,k_0} \) can be written as

\[
\frac{d\omega}{dl} V_{0,k_0} = 24 \frac{l_0^3}{k_0^3} \frac{\nu^2}{d\alpha_{k_0}} \frac{q^{k_0}}{1 - q^{2k_0}}.
\] (4.12)

Now, the amplitude of the seasonal perturbation term \( \varepsilon \) takes the form
To obtain the numerical value of the amplitude of the seasonal perturbation term, computations by formula (4.13) were performed on a computer. The results are illustrated in Fig. 3.

This figure displays the plot of the critical amplitude of the seasonal perturbation term $\varepsilon$ as a function of the frequency of the external force $\nu$. The plot displays the values of the seasonal perturbation term amplitude at which initial system (3.1.5), (3.1.6) starts to exhibit a chaotic behavior.

Typically, the process of resonance overlapping is considered in simple examples for a pair of minimal-order resonances. This is because the minimal-order resonances have the maximal amplitudes.

The system studied in this paper has a more complicated structure. It is so constructed that the number of resonances changes with the change in the frequency of the external force. This distinguishes the system under consideration from simpler examples. When $\nu < \nu_1$, there is the complete set of resonances of all orders in the system.

As the frequency $\nu$ increases, the resonances disappear one after another. It is seen from formula (3.3.18) that the frequency of the unperturbed system $\omega$ has an upper limit. It is clear that the product $k_0 \omega$ (3.5.2) also has an upper limit. For that reason, when $\nu$ is greater than this limit, the resonance of order $k_0$ disappears in the initial system. The plot displays three such values $\nu_1$, $\nu_2$, and $\nu_3$. Thus, the upper limit of the product $k_0 \omega$ determines $\nu_{k_0}$ at which the corresponding resonance disappear. As the next $\nu_{k_0}$ is crossed, the resonance $(k_0 + 1)$ becomes the minimal order resonance.

The total width of the minimal-order resonance pair is $\frac{\Delta \omega_{k_0+1}}{2} + \frac{\Delta \omega_{k_0+2}}{2}$. When $\nu_{k_0}$ is crossed, $\frac{\Delta \omega_{k_0}}{2} + \frac{\Delta \omega_{k_0+1}}{2}$ is replaced with $\frac{\Delta \omega_{k_0+1}}{2} + \frac{\Delta \omega_{k_0+2}}{2}$. Therefore, the pair of resonances of the minimal order is chosen from the remaining resonances. As the next $\nu_{k_0}$ is crossed, the pair of resonances is replaced by the next pair. In

$$
\varepsilon = \frac{1}{6} k_0^2 (k_0 - 1)^2 \frac{1}{l_0} \left[ \left( \frac{1}{d\alpha_{k_0}} \frac{q(\alpha_{k_0})^{k_0}}{1 - q(\alpha_{k_0})^{2k_0}} \right)^{1/2} + \left( \frac{1}{d\alpha_{k_0-1}} \frac{q(\alpha_{k_0-1})^{k_0-1}}{1 - q(\alpha_{k_0-1})^{2(k_0-1)}} \right)^{1/2} \right]^2.
$$

(4.13)
addition, the distance $\delta \omega$ between resonances changes. As a result, discontinuities in the plot $\varepsilon(\nu)$ at the points $\nu_1$, $\nu_2$, and $\nu_3$ appear. It is seen from this plot that, in spite of the discontinuities, the general trend is that the critical value $\varepsilon(\nu)$ increases with increasing frequency of the external force $\nu$.

Here are some features of the plot in Fig. 3:

1. Using Chirikov's method, we are able to correctly estimate the values of $\varepsilon(\nu)$ in the interval from $0.8 \nu_1$ to $2.1$ on the axis $\nu$ ($\nu_1 = 0.575$). To the left of this interval, $\varepsilon(\nu)$ is somewhat underestimated, while it is somewhat overestimated to the right of it (see Remarks 2 and 3). The critical value of $\varepsilon$ determined using Chirikov's method is located on this interval between $0.2$ and $1.0$ on the axis $\varepsilon$.

2. Due to the linearization of the perturbation term $\bar{V}$ with respect to $\zeta$ in (3.4.30), the behavior of the system for large $\zeta$ is somewhat distorted. The overestimated value of $\bar{V}$ results in the decrease of $\varepsilon$ in the plot $\varepsilon(\nu)$. This decrease applies to the interval up to the frequency of the external force $\nu$, which is $0.8 \nu_1$, $0.9 \nu_1$. The actual expected value of $\varepsilon$ is somewhat greater. In addition, the values of $\varepsilon$ on the intervals from $\nu_1$ to $0.8 \nu_2$, from $\nu_2$ to $0.8 \nu_3$, and from $\nu_3$ to $0.8 \nu_4$ are underestimated.

3. When the function $\varepsilon(\nu)$ exceeds one, the derivation of the formulas for $\varepsilon(\nu)$ becomes incorrect. This is due to the violation of Chirikov's moderate nonlinearity condition $\varepsilon << \alpha << \frac{1}{\varepsilon}$ (here, the dimensionless parameter $\alpha = \frac{I_0}{\omega_0} |\phi|$ indicates the degree of nonlinearity of the oscillations. In the case Under consideration, the parameter $\alpha$ is determined from the condition $\frac{\Omega_{ad}}{\omega_0} = (\alpha \varepsilon)^{1/2}$. This condition implies that $\alpha \propto V_0$. In turn, $\varepsilon \propto \frac{1}{V_0}$. Under this condition, the inequality $\varepsilon < \alpha$ can be written as $\frac{1}{V_0} < V_g$, and it is violated for $V_0 < 1$ and $\varepsilon > 1$). In the plot, the values that are by an order of magnitude greater than unity are shown only in order to demonstrate the behavior of the function $\varepsilon(\nu)$.

The features indicated above do not prevent the understanding of the mechanism of transition to chaos in the system under study.
5. Conclusions

Chirikov's resonance overlap method is used to investigate the transition to chaos of the system under consideration; this method enables one to estimate the interaction and overlap of the nonlinear resonances occurring in the system.

At the stage of preparing the system to the investigation by the proposed method, the following results were obtained:

– it was proved that the system is Hamiltonian;
– action–angle variables were constructed;
– the action–angle variables were represented in terms of elliptic integrals;
– the perturbation term that models the seasonal factor was thoroughly investigated;
– expansion of the perturbation term in the Fourier series was obtained;
– the resonance value of the action variable was determined.

These results suggest the conclusion that the chosen initial model is successful. As a result, both the unperturbed and perturbed parts of the initial system were thoroughly investigated.

The main results of this study are as follows:

– a contribution to the investigation of the transition to chaos for the Lotka–Volterra system and its generalizations is made;
– the main trends of the transition to chaos for the initial system are revealed;
– the critical value of the perturbation term amplitude at which the system begins to exhibit a chaotic behavior due to the appearance and interaction of resonances is determined;
– the behavior of the resonances in this system related to changes in the number of resonances depending on the changes in the frequency of the external force is described. Thus, Chirikov's resonance overlap method made it possible to investigate the mechanism of the transition to chaos in a generalization of the Lotka–Volterra system.
References


Fig. 1
Lotka–Volterra system using Chirikov's resonance overlap method

Fig. 2
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