Tree Cover of Graphs

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Abstract

Let $G$ be a graph and $\mathcal{F}_G = \{G_1, G_2, G_3, \ldots, G_n\}$ be a collection of subgraphs of $G$ where $G_i$ is a tree for all $i = 1, 2, \ldots, n$. If for every edge $e \in E(G)$, there exists $G_i \in \mathcal{F}_G$ such that $e \in E(G_i)$, then $\mathcal{F}_G$ is a tree cover of $G$. The tree covering number of $G$ is given by

$$t_c(G) = \min\{|\mathcal{F}_G| : \mathcal{F}_G \text{ is a tree cover of } G\}.$$

This paper characterizes graphs with tree covering number equal to some nonnegative integers in relation to the size and the order of the graph. Moreover, the tree coverings and the tree covering numbers of the cycle, fan, wheel, and a cyclic graph in general is obtained. We also introduced here a concept of cycle derivative and obtain results on the tree covering number with respect to this graph operation.

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1 Introduction

Let $G$ be a graph. A tree of a graph $G$ is a connected acyclic subgraph of $G$. A collection $\mathcal{F}_G = \{G_1, G_2, G_3, \ldots, G_n\}$ of subgraphs of $G$ is a tree cover of $G$ if $G_i$ is a tree for all $i = 1, 2, \ldots, n$ and for every edge $e \in E(G)$, there exists
$G_i \in \mathcal{F}_G$ such that $e \in E(G_i)$. The tree covering number of $G$, denoted by $t_c(G)$, is given by

$$t_c(G) = \min\{|\mathcal{F}_G| : \mathcal{F}_G \text{ is a tree cover of } G\}.$$

We first introduce here the following definition of the join of graphs since the fan and the wheel are expressed as the join of graphs, to ease our approach in the proof of our results.

**Definition 1.1** [4] Let $G$ and $H$ be vertex disjoint graphs. The *join* $G \oplus H$ of $G$ and $H$ has vertex-set $V(G \oplus H) = V(G) \cup V(H)$ and edge-set

$$E(G \oplus H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$$

The following section establishes some basic results and properties of tree covering of graphs. Some characterizations are also obtained here relating the size and the order of the graph.

## 2 Basic Results and Some Characterizations

We provide in this section some basic properties of the tree covering number of graphs and some characterizations of graphs with tree covering number equal to zero, tree covering number equal to one, and graphs with tree covering number equal to its size. Clearly, for any graph $G$, $t_c(G) \geq 0$.

The sharpness of this upper bound is established in the following theorem.

**Theorem 2.1** Let $G$ be a graph of order $n$. Then $t_c(G) = 0$ if and only if $G \cong K_n$.

**Proof:** Assume that $t_c(G) = 0$. Then $|E(G)| = 0$. Since $|V(G)| = n$, it follows that $G \cong K_n$.

Conversely, assume that $G \cong K_n$. By definition, $t_c(G) \leq 0$. Combining this with the inequality $t_c(G) \geq 0$, we have $t_c(G) = 0$. \[\square\]

The above theorem tells us that the tree covering number is a positive definite operator in graphs. Also, this asserts that if $G$ is nonempty, then $t_c(G) \geq 1$.

The next result characterizes all nontrivial connected graphs with tree covering number equal to one.

**Theorem 2.2** Let $G$ be a nontrivial connected graph of order $n$. Then $t_c(G) = 1$ if and only if $G$ is a tree.
Proof: Assume $t_c(G) = 1$. Then there exists a covering $\mathcal{F}_G$ of $G$ such that $|\mathcal{F}_G| = 1$. Let $\mathcal{F}_G = \{T\}$. Since $G$ is connected and nontrivial, $G = T$. Thus, $G$ is a tree.

Conversely, assume that $G$ is a tree. Let $\mathcal{F}_G = \{G\}$. Then $\mathcal{F}_G$ is a tree cover of $G$. By definition, $t_c(G) \leq |\mathcal{F}_G| = 1$. Since $G$ is not the empty graph, a direct consequence of Theorem 2.1 asserts that $t_c(G) \geq 1$. Accordingly, $t_c(G) = 1$. $\square$

The following is a direct consequence of the above theorem.

**Corollary 2.3** If $G$ is a nontrivial connected acyclic graph, then $t_c(G) = 1$.

**Corollary 2.4** For every positive number $n \geq 2$, the following hold:

1. $t_c(P_n) = 1$
2. $t_c(T_n) = 1$
3. $t_c(S_n) = t_c(K_{1,n}) = 1$.

Proof: Since $T_n$, $P_n$, and $S_n$ for $n \geq 2$ are nontrivial connected acyclic graphs, then the results follow from the preceding corollary. $\square$

The following result characterizes all nontrivial connected graphs with tree covering number equal to its size.

**Theorem 2.5** Let $G$ be a nontrivial connected graph of order $n \geq 2$. Then, $t_c(G) = |E(G)|$ if and only if $G \cong K_2$.

Proof: Assume that $t_c(G) = |E(G)|$. Since $G$ is connected, $|V(G)| = 2$. Accordingly, $G \cong K_2$.

Conversely, assume that $G \cong K_2$. Then $t_c(G) = t_c(K_2) = 1$, by Theorem 2.2. Since $|E(G)| = 1$, we must have $t_c(G) = |E(G)|$. $\square$

The next section investigates the tree covering and the tree covering number of cycles, fans, and wheels.

### 3 Tree Cover of Cycles, Fans, and Wheels

We establish the tree covering number of the cycle in the following theorem.

**Theorem 3.1** For every positive integer $n \geq 3$, $t_c(C_n) = 2$. 
Proof: The contrapositive of the implication in Theorem 2.2 asserts that \( t_c(C_n) \geq 2 \). Consider the cycle \( C_n = [u_1, u_2, u_3, \ldots, u_n, u_1] \). Let \( F = \{G_1, G_2\} \), where \( G_1 = [u_1, u_2, u_3, \ldots, u_n] \) and \( G_2 = [u_n, u_1] \). Then \( G_1 \) and \( G_2 \) are trees. Hence, \( F \) is a tree cover of \( C_n \). Thus, \( t_c(C_n) \leq |F| = 2 \). Accordingly, \( t_c(C_n) = 2 \).

The tree covering number of the fan is established in the following theorem.

**Theorem 3.2** For every positive integer \( n \geq 2 \), \( t_c(F_n) = 2 \).

Proof: For \( n \geq 2 \), \( t_c(F_n) \geq 2 \), by Theorem 2.2. Note that by definition, \( F_n = P_n + K_1 \). Let \( P_n = [u_1, u_2, \ldots, u_n] \) and \( V(K_1) = \{v\} \). Then \( vu_i \in E(F_n) \) for all \( i = 1, 2, 3, \ldots, n \). Consider \( F = \{G_1, G_2\} \), where \( G_1 = P_n \) and \( G_2 \) is the star \( K_{1,n} = \{v\} \oplus K_n \) with \( V(K_n) = \{u_1, u_2, \ldots, u_n\} \). Then \( G_1 \) and \( G_2 \) are trees. Moreover, \( F \) is a tree cover of \( F_n \). Thus, \( t_c(F_n) \leq |F| = 2 \). Accordingly, \( t_c(F_n) = 2 \).

Next, we establish the tree covering number of the wheel \( W_n \).

**Theorem 3.3** For every positive integer \( n \geq 3 \), \( t_c(W_n) = 2 \).

Proof: By Theorem 2.2, \( t_c(W_n) \geq 2 \). By definition, \( W_n = C_n + K_1 \). Let \( C_n = [u_1, u_2, u_3, \ldots, u_n, u_1] \) and \( V(K_1) = \{v\} \). Then \( vu_i \in E(W_n) \) for all \( i = 1, 2, 3, \ldots, n \). Consider \( F = \{G_1, G_2\} \), where \( G_1 = [u_1, u_2, u_3, \ldots, u_n, v] \) and \( G_2 = G \setminus E(G_1) \). Then, \( F \) is a tree cover of \( W_n \). Thus, \( t_c(W_n) \leq 2 \). Accordingly, \( t_c(W_n) = 2 \).

We establish our result in on cyclic graphs in the following section.

## 4 Tree Cover of Cyclic Graphs

We present new definitions here that are useful in obtaining the results on the tree cover and tree covering number of cyclic graphs.

**Theorem 4.1** Let \( G \) be a unicyclic graph. Then \( t_c(G) = 2 \).

Proof: Let \( G \) be a unicyclic graph and \( C_t = [u_1, u_2, u_3, \ldots, u_t, u_1] \) be the unique cycle in \( G \) of order \( t \). Consider \( F = \{G_1, G_2\} \), where \( G_1 = G \setminus \{u_1, u_t\} \) and \( G_2 = [u_1, u_t] \). Then \( G_1 \) and \( G_2 \) are trees. Moreover, \( F \) is a tree cover of \( G \). Thus, \( t_c(G) \leq 2 \). By Theorem 2.2, \( t_c(G) \geq 2 \). Accordingly, \( t_c(G) = 2 \). \( \square \)
**Definition 4.2** Let $G$ be a nontrivial connected graph. A cycle $C_t$ in $G$ is said to be minimal (minimal cycle or $m_t$-cycle) if $C_t$ does not contain any cycle of order less than $t$. The $m_t$-cycle derivative of $G$, denoted by $m_tG'$ is the graph obtain from $G$ by taking the $m_t$-cycles of $G$ as vertices in $m_tG'$ and two vertices in $m_tG'$ are adjacent if and only if the two $m_t$-cycles in $G$ corresponding to these vertices have an edge in common.

**Theorem 4.3** Let $G$ be a nontrivial connected graph. If $m_tG' = K_1$ then $t_c(G) = 2$.

**Proof**: Assume that $m_tG' = K_1$. Then $G$ is unicyclic. Thus, by Theorem 4.1, $t_c(G) = 2$. \[\square\]

**References**


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