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Fuzzy (φ, ψ)-Contractive Mapping and Fixed Point Theorem

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Abstract

In this paper, we define a new contractive mapping called fuzzy (φ, ψ)-contraction in fuzzy metric space and prove a fixed point theorem for this type of mapping. Finally, an example is given to illustrate the main result of this paper.

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1. Introduction and Preliminaries

The concept of fuzzy metric spaces was defined in different ways [1, 3, 11]. Grabiec [9] presented a fuzzy version of Banach contraction principle in a fuzzy metric space of Kramosi and Michalek's sense. Fang [4] proved some fixed point theorems in fuzzy metric spaces, which improve, generalize, unify and extend some main results of Edelstein [2], Istratescu [10], Sehgal and Bharucha-Reid [14].

In order to obtain a Hausdorff topology, George and Veeramani [6, 7] modified the concept of fuzzy metric space due to Kramosil and Michalek [12]. Many fixed point theorems in complete fuzzy metric spaces in the sense of George and Veeramani [6,7] have been obtained. Gregori and Sapena [8] proved that each fuzzy contractive mapping has a unique fixed point in a complete GV fuzzy metric space in which fuzzy contractive sequences are Cauchy.

In 2008, Mihet gave a mapping called fuzzy ψ-contraction [13]. Let $(X, M, *)$ be a fuzzy metric space and let $f : X \to X$ be a mapping. The mapping f is called a fuzzy ψ-contraction if the following holds:

if $M(x, y, t) > 0$, then $M(fx, fy, t) \geq \psi(M(x, y, t))$.

Then the author proved a fixed point theorem for this type of mapping in non-Archimedean fuzzy metric spaces as follows.

Theorem 1.1 ([13]). *Let $(X, M, *)$ be an M-complete non-Archimedean fuzzy metric space and $f : X \to X$ be a fuzzy ψ-contractive mapping. If there exists $x \in X$ such that $M(x, fx, t) > 0$ for all $t > 0$, thenf has a fixed point.*

Recently, Wang [16] improves the result of Mihet from non-Archimedean fuzzy metric space to fuzzy metric space as follows.

Theorem 1.2 ([16]). *Let $(X, M, *)$ be an M-complete fuzzy metric space in the sense of Kramosil and Michalek with $*$ positive, and $f : X \to X$ be a fuzzy ψ-contractive mapping. If there exists $x \in X$ such that $M(x, fx, t) > 0$ for all $t > 0$, thenf has a fixed point.*

In this paper, we give a new mapping called fuzzy (φ, ψ)-contraction and prove a fixed point theorem for this type of mapping in fuzzy metric spaces. Then an example will be given to illustrate the main result of this paper.

The following are some necessary concepts and lemma which will be used in next section.

Definition 1.1([6]). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ satisfies the following conditions:

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0,1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Two typical examples of the continuous t-norm are $a *_1 b = ab$ and $a *_2 b = \min\{a, b\}$ for all $a, b \in [0,1]$.

Definition 1.2 (Kramosil and Michalek [12]). A fuzzy metric space (in the sense of Kramosil and Michalek) is a triple $(X, M, *)$, where X is a nonempty set, $*$ is a continuous t-norm and $M : X^2 \times [0, \infty)$ is a mapping, satisfying the following:

(KM-1) $M(x, y, 0) = 0$ for all $x, y \in X$,

(KM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$, (KM-3) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and all $t > 0$,

(KM-4) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous for all $x, y \in X$,

(KM-5) $M(x, y, t+s) \geq M(x, z, t) * M(y, z, s)$ for all $x, y, z \in X$ and all $t, s > 0$.

If the triangular inequality (KM-5) is replaced by

$$\text{(NA)} \quad M(x, y, t) \geq M(x, z, t) * M(z, y, t) \ \text{ for all } x, y, z \in X \text{ and } t > 0,$$

then $(X, M, *)$ is called a non-Archimedean fuzzy metric space [15]. It is easy to check that the triangle inequality (NA) implies (KM-5), that is, every non-Archimedean fuzzy metric space is itself a fuzzy metric space.

In Definition 1.2, if M is a fuzzy set on $X^2 \times (0, \infty)$ and (KM-1), (KM-2), (KM-4) are replaced with the following (GV-1), (GV-2), (GV-4), respectively, then $(X, M, *)$ is called a fuzzy metric space in the sense of George and Veeramani [6]:

(GV-1) $M(x, y, t) > 0$ for all $t > 0$ and all $x, y \in X$,

(GV-2) $M(x, y, t) = 1$ for some $t > 0$ if and only if $x = y$,

(GV-4) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

Lemma 1.1 ([9]). Let $(X, M, *)$ be a fuzzy metric space in the sense of GV. Then $M(x, y, \cdot)$ is nondecreasing for all $x, y \in X$.

Definition 1.3 (George and Veeramani [6]). Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is called an M-Cauchy sequence, if for each $\epsilon \in (0, 1)$ and $t > 0$ there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $m, n \geq n_0$. The fuzzy metric space $(X, M, *)$ is called M-complete if every M-Cauchy sequence is convergent.

2. Main results

Let $\mathbf{\Phi_w}$ denote the set of all functions $\varphi : (0, \infty) \to (0, \infty)$ satisfying the condition

for each $t > 0$, there exists $r \geq t$ such that $\lim_{n\to\infty} \varphi^n(r) = 0$.

Lemma 2.1 ([5]). *Let $\varphi \in \mathbf{\Phi_w}$. The for each $t > 0$, there exists $r \geq t$ such that $\varphi(r) < t$.*

The following lemma is directly from [5].

Lemma 2.2. *Let $(X, M, *)$ be a fuzzy metric space and $\varphi \in \mathbf{\Phi_w}$. If $\sup_{t>0} M(x, y, t) = 1$ and $M(x, y, \varphi(t)) \geq M(x, y, t)$ for each $t > 0$, then $x = y$.*

Let $\mathbf{\Psi}$ denote the set of all functions $\psi : [0,1] \to [0,1]$ satisfying the condition: ψ is non-decreasing and left continuous, and for each $t \in (0,1)$, $\psi(t) > t$.

Obviously, for each $t \in (0,1)$, one has $\lim_{n\to\infty} \psi^n(t) = 1$.

Let $(X, M, *)$ be a fuzzy metric space in the sense of Kramosil and Michalek and $f : X \to X$ be a mapping. The mapping f is called a (φ, ψ)-contraction if the following holds:

$M(x, y, t) > 0$ implies that $M(fx, fy, \varphi(t)) \geq \psi(M(x, y, t))$.

Next we state the main result of this paper.

Theorem 2.1. *Let $(X, M, *)$ be an M-complete fuzzy metric space in the sense of GV with $*$ positive, and $f : X \to X$ be a fuzzy (φ, ψ)-contractive mapping, where $\varphi \in \mathbf{\Phi_w}$ and $\psi \in \mathbf{\Psi}$. If there is $x \in X$ such that $M(x, fx, t) > 0$ for all $t > 0$, then f has a fixed point in X. If for each $x, y \in X$, $\sup_{t>0} M(x, y, t) = 1$, then f has a unique fixed point in X.*

Proof. Define the sequence $\{x_n\}_{n=0}^{\infty}$ by $x_0 = x$ and $x_n = f^n x$ for each $n \in \mathbb{N}$. We first show that for each $n \in \mathbb{N}$,

$$M(x_n, x_{n+1}, t) \geq \psi(M(x_{n-1}, x_n, t)).$$

In fact, Lemma 2.1 shows that for each $t > 0$, there exists $r \geq t$ such that $\varphi(r) < t$. So we have

$$M(x_n, x_{n+1}, t) \geq M(x_n, x_{n+1}, \varphi(r)) \geq \psi(M(x_{n-1}, x_n, r)) \geq \psi(M(x_{n-1}, x_n, t)).$$

Further, by induction we have, for each $n \in \mathbb{N}$,

$$M(x_n, x_{n+1}, t) \geq \psi^n(M(x_0, x_1, t)).$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1, \quad \forall t > 0,$$

which implies that

$$\lim_{n\to\infty} M(x_n, x_{n+k}, t) = 1, \quad \forall t > 0.$$

Let $t > 0$. Define $a_n = \inf_{k\geq 1} M(x_n, x_{n+k}, t)$. Then we have

$$\begin{aligned} M(x_n, x_{n+k}, t) &\geq \psi(M(x_{n-1}, x_{n-1+k}, t)) \\ &\geq M(x_{n-1}, x_{n-1+k}, t). \end{aligned}$$

Taking inf in k from both sides of above inequality, we have

$$a_n = \inf_{k\geq 1} M(x_n, x_{n+k}, t) \geq \inf_{k\geq 1} M(x_{n-1}, x_{n-1+k}, t) = a_{n-1}.$$

Thus there exists $l \in [0,1]$ such that $a_n \to l$ as $n \to \infty$. We show that $l > 0$. For any given $\epsilon \in (0, \frac{1}{2})$, by the definition of a_n, there exists $k \in \mathbb{N}$ such that $\epsilon + a_n > M(x_n, x_{n+k}, t)$. Fixing k and letting $n \to \infty$, we have

$$\epsilon + l \geq 1,$$

which implies that $l > 0$. Next we show that $l = 1$. Since ψ is left continuous, we have $l \geq \psi(l)$. Thus $l = 1$.

For any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $a_n > 1 - \epsilon$ for all $n > N$, which implies that $M(x_n, x_{n+k}, t) > 1 - \epsilon$ for all $n > N$ and $k \in \mathbb{N}$. Therefore $\{x_n\}$ is a Cauchy sequence. Since X is complete, there is $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

We prove that x^* is the fixed point of f. From $M(x^*, fx^*, t) \geq M(x^*, x_n, t/2) * M(x_n, fx^*, t/2) \geq M(x^*, x_n, t/2) * \psi(M(x_{n-1}, x^*, t/2))$. Letting $n \to \infty$, since $*$ is continuous, we have $M(x^*, fx^*, t) \geq 1$, which implies that $x^* = fx^*$. So x^* is a fixed point of f.

Finally, we prove that if for each $x, y \in X$, $\sup_{t>0} M(x, y, t) = 1$, then f has a unique fixed point in X. Assume that $x' \in X$ is another fixed point of f. Then we have

$$M(x^*, x', \varphi(t)) = M(fx^*, fx', \varphi(t)) \geq \psi(M(x^*, x', t)) \geq M(x^*, x', t).$$

By Lemma 2.2, we have $x^* = x'$. Therefore, x^* is the unique fixed point of f. This complete the proof. □

Since each non-Archimedean fuzzy metric space itself is a fuzzy metric space, the following corollary holds.

Corollary 2.1. *Let $(X, M, *)$ be an M-complete non-Archimedean fuzzy metric space with $*$ positive, and $f : X \to X$ be a fuzzy (φ, ψ)-contractive mapping, where $\varphi \in \mathbf{\Phi_w}$ and $\psi \in \mathbf{\Psi}$. If there is $x \in X$ such that $M(x, fx, t) > 0$ for all $t > 0$, then f has a fixed point in X. If for each $x, y \in X$, $\sup_{t>0} M(x, y, t) = 1$, then f has a unique fixed point in X.*

Finally, we give an example to illustrate Theorem 2.1 as follows.

Example 2.1. Let $X = [0, \infty)$ and $M(x, y, t) = \frac{\min\{x^2, y^2\}}{\max\{x^2, y^2\}}$ for all $x, y \in X$ and $t > 0$. Then $(X, M, *_P)$ is a complete fuzzy metric space. Let $fx = \sqrt{x}$ for all $x \in X$. Define two functions $\varphi(t) = t/2$ and $\psi(t) = \sqrt{t}$ for all $t > 0$. Then it is easy to see that $\varphi \in \mathbf{\Phi_w}$ and $\psi \in \mathbf{\Psi}$. Moreover, for all $x, y \in X$ and $t > 0$, one has

$$M(fx, fy, \varphi(t)) = \frac{\min\{\sqrt{x}, \sqrt{y}\}^2}{\max\{\sqrt{x}, \sqrt{y}\}^2} = \sqrt{\frac{\min\{x, y\}^2}{\max\{x, y\}^2}} = \psi(M(x, y, t)).$$

So f is a fuzzy (φ, ψ)-contraction. All the conditions of Theorem 2.1 are satisfied. By Theorem 2.1 the mapping f has a fixed point in X. Note that

$\sup_{t>0} M(x, y, t) \neq 1$ in this example. So by Theorem we can not conclude that f has the unique fixed point. In fact, f has two fixed point $x^* = 0, 1$.

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