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Common Fixed Point for a Pair of Weak** Commuting Maps

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Abstract

As a weaker form of commuting mappings, Sessa (1982) defined weakly commuting maps and Pathak (1986) introduced the notions of weak* and weak** commuting maps. Both the notions are equivalent to the notion of Sessa if the involved maps are idempotent. The present paper aims at proving a common fixed point theorem for weak** commuting maps using a generalized rational inequality under a convergence criterion which relaxes the completeness of the underlying space. Our result generalizes the result of Lohani and Badshah (1995).

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1. Introduction

Let (X, d) denote a metric space. If f is a self-map on X, we write f^n for its nth iterate. Self-map f is idempotent if $f^2 = f$. A point $p \in X$ is a *coincidence point* of self-maps f and g if $fp = gp$, the common image $fp = gp = q$ being a point of coincidence for them. Self-maps f and g are commuting if $fg = gf$.

Definition 1.1. (Sessa, [5]) Self-maps f and g are weakly commuting if

$$d(fgx, gfx) \leq \ d(fx, gx) \quad \text{for all} \quad x \in X. \tag{1.1}$$

Definition 1.2. (Pathak, [3]) Self-maps f and g are weak* commuting if

$$d(fgx, gfx) \leq \ d(f^2x, g^2x) \quad \text{for all} \quad x \in X. \tag{1.2}$$

We see that (1.1) and (1.2) are trivial whenever $fgx = gfx$ for all $x \in X$. That is, commutativity implies weak* commutativity and weak commutativity. However there can be a noncommuting pair which is weakly commuting and weak* commuting as revealed by the following example:

Example 1.1. Let $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define

$$fx = ax, \ gx = \frac{bx}{bx+1} \quad \text{for all} \quad x \in X,$$

where $0 < a, b \leq \frac{1}{2}$ with $b \leq a$. Then $f(x) \geq g(x)$ for all $x \in X$ and

$$d(fgx, gfx) = \frac{abx}{bx+1} - \frac{abx}{abx+1} \leq a^2x - \frac{bx}{b^2x + bx + 1} = d(f^2x, g^2x) \text{ for all } x \in X.$$

Thus f and g are weak* commuting. Again

$$d(f^2x, g^2x) = a^2x - \frac{bx}{b^2x + bx + 1} \leq ax - \frac{bx}{bx+1} = d(fx, gx) \text{ for all } x \in X$$

and so (1.1) holds good. That is f and g are weakly commuting.

Writing $a = b = \dfrac{1}{2}$ in this, we get Example 2.3 of Pathak [3].

Though weak* commutativity and weak commutativity are weaker than commutativity, it has been not known so far whether these two versions are independent (of each other). We first demonstrate that these two notions are in fact independent.

Example 1.2. Let $X = [0, \infty)$ with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define $f, g : X \to X$ by

$$fx = \begin{cases} 0 & \text{if } x < 1 \\ \dfrac{x}{2} & \text{if } x \geq 1 \end{cases} \quad \text{and} \quad gx = \begin{cases} 0 & \text{if } x < 1 \\ \dfrac{x}{3} & \text{if } x \geq 1. \end{cases}$$

Then

$$d(fx,gx)=\begin{cases}0 & \text{if } x<1\\ \dfrac{x}{6} & \text{if } x\geq 1,\end{cases}\quad d(f^2x,g^2x)=\begin{cases}0 & \text{if } x<3\\ \dfrac{x}{6} & \text{if } x\geq 3\end{cases}$$

and

$$d(fgx,gfx)=\begin{cases}0 & \text{if } x<2\\ \dfrac{x}{6} & \text{if } 2\leq x<3\\ 0 & \text{if } x\geq 3.\end{cases}$$

This shows that (1.1) holds good and f and g are weakly commuting. However, (1.2) fails when $2 \leq x < 3$ so that (f, g) is not weak* commuting.

Example 1.3. Let $X = \left\{ \left(\pm\frac{1}{2}, 0\right), \left(0, \pm\frac{1}{2}\right) \right\}$ with usual metric $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$ for all $\mathbf{x}, \mathbf{y} \in X$. Define $f, g : X \to X$ by

$$f\left(\frac{1}{2},0\right)=f\left(0,\frac{1}{2}\right)=\left(\frac{1}{2},0\right),f\left(-\frac{1}{2},0\right)=f\left(0,-\frac{1}{2}\right)=\left(0,\frac{1}{2}\right);$$

$$g\left(\frac{1}{2},0\right)=g\left(0,-\frac{1}{2}\right)=\left(\frac{1}{2},0\right),g\left(-\frac{1}{2},0\right)=g\left(0,\frac{1}{2}\right)=\left(0,\frac{1}{2}\right).$$

A routine computation yields that (1.2) holds good, that is f and g are weak* commuting. But for $\mathbf{x} = \left(-\frac{1}{2}, 0\right)$, we see that $d(f\mathbf{x}, g\mathbf{x}) = 0 < \frac{1}{\sqrt{2}} = d(fg\mathbf{x}, gf\mathbf{x})$, showing that f and g are not weakly commuting.

Thus weak* commutativity and weak commutativity are independent notions, but are equivalent if both f and g are idempotent.

With some additional conditions on weak* commutativity, Pathak [4] defined weak** commuting maps as follows.

Definition 1.3. (Pathak, [4]) Self-maps f and g are weak** commuting if

$$f(X) \subset g(X) \tag{1.3}$$

and

$$\begin{aligned} d(f^2g^2x, g^2f^2x) &\leq d(f^2gx, gf^2x)\\ &\leq d(g^2fx, fg^2x)\\ &\leq d(fgx, gfx)\\ &\leq d(f^2x, g^2x) \text{ for all } x \in X. \end{aligned} \tag{1.4}$$

With his notion, Pathak [4] proved a common fixed point theorem for four continuous maps on compact metric space. Lohani and Badshah [2] obtained the following result for a pair of self-maps on complete metric space:

Theorem 1.1. *Let f and g be self-maps on complete metric space X satisfying the inclusion*

$$f(X) \subset g(X) \tag{1.5}$$

and either of the rational inequalities

$$d(f^2g^2x, g^2f^2y) \leq \frac{ad(f^2y, g^2f^2y)[1 + d(g^2x, f^2g^2x)]}{1 + d(g^2x, f^2y)} + \ cd(g^2x, f^2y) \tag{1.6}$$

or

$$d(f^2f^2x, g^2f^2y) \leq \frac{bd(f^2y, f^2g^2x)[1 + d(g^2x, g^2f^2y)]}{1 + d(g^2x, f^2y)} + \ cd(f^2x, f^2y) \tag{1.7}$$

for all $x, y \in X$, where $a, c \geq 0$ with $a + c < 1$ and $b, c \geq 0$ with $b + c < 1$ respectively. If g is continuous and (f, g) is weak** commuting, then f and g will have a unique common fixed point.

We prove a generalization of Theorem A under a unified general form of (1.6) and (1.7) by relaxing the completeness of the space X, where either f or g is continuous.

Given $x_0 \in X$ and f and g self-maps on X, we define the associated sequence $(x_n)_{n=1}^{\infty}$ at x_0 by

$$f^2(g^2f^2)^{n-1}x_0 = x_{2n-1}, (g^2f^2)^n x_0 = x_{2n}, \text{ for } n = 1, 2, 3, ..., \tag{1.8}$$

2. Main Result

The following is our main result which was presented in the 22nd Congress of Andhra Pradesh Society for Mathematical Sciences recently held at Hyderabad, Andhra Pradesh, India (December 13-15, 2013).

Theorem 2.1. *Let f and g be self-maps on X satisfying the inclusion* (1.5) *and the rational inequality*

$$\begin{aligned} d(f^2g^2x, g^2f^2y) \leq a &\left\{ \frac{d(f^2y, g^2f^2y)[1 + pd(g^2x, f^2g^2x)]}{1 + qd(g^2x, f^2y)} \right\} \\ &+ b \left\{ \frac{d(f^2y, f^2g^2x)[1 + pd(g^2x, g^2f^2y)]}{1 + qd(g^2x, f^2y)} \right\} \\ &+ \ cd(g^2x, f^2y) \quad \text{for all} \quad x, y \in X, \end{aligned} \tag{2.1}$$

where $a, b, c \geq 0$ with $a + b + c < 1$ and $p, q > 0$ with $p \leq q$. Suppose that there a point $x_0 \in X$ such that

(a) *the associated sequence $(x_n)_{n=1}^{\infty}$ at x_0 converges to some $z \in X$.*

If either f or g is continuous, then f^2 and g^2 have a common fixed point z. Further if (f, g) is weak** commuting, then z will be a unique common fixed point of f and g.

Proof. Let $x_0 \in X$ be such that (a) holds good. That is

$$\lim_{n\to\infty} f^2(g^2f^2)^{n-1}x_0 = \lim_{n\to\infty}(g^2f^2)^n x_0 = z. \tag{2.2}$$

Suppose that g is continuous. Then from (2.2), we get

$$g^2\left(\lim_{n\to\infty} f^2(g^2f^2)^{n-1}x_0\right) = \lim_{n\to\infty} g^2\left(f^2(g^2f^2)^{n-1}x_0\right)$$

or

$$g^2z = z \tag{2.3}$$

Writing $x = z$ and $y = (g^2f^2)^n x_0$ in (2.1), we see that

$$\begin{aligned} d\left(f^2g^2z, (g^2f^2)^{n+1}x_0\right) &= d\left(f^2g^2z, g^2f^2(g^2f^2)^n x_0\right) \\ &\le a\left\{\frac{d(f^2(g^2f^2)^n x_0, g^2f^2(g^2f^2)^n x_0)[1+pd(g^2z, f^2g^2z)]}{1+qd(g^2z, f^2(g^2f^2)^n x_0)}\right\} \\ &\quad + b\left\{\frac{d(f^2(g^2f^2)^n x_0, f^2g^2z)[1+pd(g^2z, g^2f^2(g^2f^2)^n x_0)]}{1+qd(g^2z, f^2(g^2f^2)^n x_0)}\right\} \\ &\quad + cd(g^2z, f^2(g^2f^2)^n x_0). \end{aligned} \tag{2.4}$$

Applying the limit as $n \longrightarrow \infty$ and using (2.2) and (2.3), this gives

$$d(f^2z, z) \le a\left\{\frac{d(z,z)[1+pd(z,f^2z)]}{1+qd(z,z)}\right\} + b\left\{\frac{d(z,f^2z)[1+pd(z,z)]}{1+qd(z,z)}\right\} + cd(z,z)$$

or $d(f^2z, z) \le cd(z, f^2z)$ so that $d(f^2z, z) = 0$, since $c < 1$. Thus

$$f^2z = g^2z = z \tag{2.5}$$

Now suppose that f is continuous. Then from (2.2), we get

$$f^2\left(\lim_{n\to\infty}(g^2f^2)^n x_0\right) = \lim_{n\to\infty}\left(f^2(g^2f^2)^n x_0\right)$$

or

$$f^2z = z \tag{2.6}$$

so that $g^2f^2z = g^2z$. Writing $x = f^2(g^2f^2)^n x_0$ and $y = z$ in (2.1), we see that

$$\begin{aligned} d\left(f^2g^2f^2(g^2f^2)^n x_0, g^2f^2z\right) &= d\left(f^2(f^2g^2)^{n+1}x_0, g^2f^2z\right) \\ \le a&\left\{\frac{d(f^2z, g^2f^2z)[1+pd(g^2f^2(g^2f^2)^n x_0, f^2g^2f^2(g^2f^2)^n x_0)]}{1+qd(g^2f^2(g^2f^2)^n x_0, f^2z)}\right\} \\ + b&\left\{\frac{d(f^2z, f^2g^2f^2(g^2f^2)^n x_0)[1+pd(g^2f^2(g^2f^2)^n x_0, g^2f^2z)]}{1+qd(g^2f^2(g^2f^2)^n x_0, f^2z)}\right\} \\ &\quad + cd(g^2f^2(g^2f^2)^n x_0, f^2z). \end{aligned}$$

Applying the limit as $n \longrightarrow \infty$ and using (2.2) and (2.6), this gives

$$\begin{aligned} d\left(z, g^2z\right) &\le a\left\{\frac{d(z,g^2z)[1+pd(z,z)]}{1+qd(z,z)}\right\} \\ &\quad + b\left\{\frac{d(z,z)[1+pd(z,g^2z)]}{1+qd(z,z)}\right\} + cd(z,z), \end{aligned}$$

which simplifies as $d(z, g^2z) \leq ad(z, g^2z)$ so that $d(z, g^2z) = 0$ or $g^2z = z$, since $a < 1$. This proves (2.5).

In view of weak** commutativity of (f, g), (2.5) gives $f^2gz = gz$ and $g^2fz = fz$. This and (2.1) with $x = y = z$ imply that

$$\begin{aligned} d(z, fz) = d(f^2g^2z, g^2f^2z) &\leq a\left\{\frac{d(g^2fz, g^2f^2(fz))[1 + pd(f^z, f^2g^2z)]}{1 + qd(f^2z, g^2fz)}\right\} \\ &\quad + b\left\{\frac{d(g^2fz, g^2f^2(fz))[1 + pd(f^z, f^2g^2z)]}{1 + qd(f^2z, g^2fz)}\right\} \\ &\quad + bd(f^2z, g^2fz) \\ &= bd(z, fz) \end{aligned}$$

so that $d(z, fz) = 0$ or $fz = z$, since $b < 1$. Similarly we can show that $gz = z$, and hence z is a common fixed point of f and g.

The uniqueness of the common fixed point follows easily from (2.1). $\square$

Remark 2.1. Given $x_0 \in X$, it can be shown as in the the proof of Theorem 1.1 that the associated sequence $(x_n)_{n=1}^{\infty}$ at x_0 with the choice (1.8) is a Cauchy sequence, where f and g satisfy (2.1). Therefore if $x_n \to z$ for some $z \in X$ as $n \to \infty$, provided X is complete.

Writing $p = q = 1$ in (2.1), we get (1.6) and (1.7) as particular cases when $b = 0$ and $a = 0$ respectively. It may be noted that Theorem 1.1 uses the continuity of g only. Thus the results of Lohani and Badshah [2] follow as particular cases from Theorem 2.1.

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