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Monotonic Interpolating Curves by Using Rational Cubic Ball Interpolation

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Abstract

This paper discusses the monotonicity preserving of monotone data by using rational cubic Ball interpolant with four parameters. The sufficient condition for the monotonicity of the rational interpolant will be derived on two of the parameters meanwhile the remaining two are free parameters that can be used to modify the final shape of the monotonic interpolating curves. The degree smoothness achieved is C^1. The first derivative is estimated by using arithmetic mean method (AMM) and geometric mean method (GMM). Several numerical results will be presented including comparison with existing scheme. From the numerical results, rational cubic Ball gives visually pleasing results.

Keywords: Rational interpolant, cubic Ball, monotone, sufficient, parameters, continuity, numerical

1. Introduction

In computer graphics and scientific visualization, the display curves and surfaces (or images) are an important and main task for the designer and expertise. Usually the given data has its own characteristics such as positivity, monotonicity and convexity. For example if the data is monotone (the first derivative have either positive value or negative value), then the methods that have been used to display the curves or surfaces must be able to retain the original shape of the data namely monotone. For examples the dose-response curves and surfaces in biochemistry and pharmacology are other examples in which the monotonicity exists in the data sets Beliakov [1]. This paper is a continuation of our previous research in Karim [2, 3, 4] and Karim and Kong [5].

There exist many methods that can be used for monotonicity preserving. Two early papers are Fritsch and Carlson [6] and Dougherty et al. [7]. Both methods require the modification(s) of the first derivative if the monotonicity of the data is not preserves. Karim and Kong [5], Sarfraz et al. [8, 9, 10], Sarfraz [11, 12] and Hussain and Sarfraz [13] have discussed the shape preserving interpolation by using various types of polynomial spline and rational cubic spline interpolant. All the methods do not require any modification of the first derivative in order to maintain the monotonicity of the given monotone data sets.

Motivated by work of Karim [2, 3, 4], in this paper the author will extend the rational cubic Ball with four parameters that has been initiated by Karim [2] for monotonicity preserving. The sufficient condition for the rational cubic Ball to be monotone on the entire given interval will be derived on two parameters while the remaining two parameters are free parameters that can be used to refine the final shape of the monotonic interpolating curves. The sufficient conditions provide visually pleasing monotonic interpolating curves with two free parameters α_i, δ_i that can be further utilized by the user.

These papers have the following contribution to the field of scientific visualization and computer graphics:

1. In this paper the rational cubic Ball interpolant (cubic/cubic) initiated by Karim [2] has been used for monotonicity preserving while Hussain and Sarfraz [13] have used rational cubic spline (cubic/cubic) with four parameters too. The sufficient conditions for monotonicity-preserving can be used to generate the monotonic interpolating curves. But by using rational cubic Ball interpolation the computation may be lower than the rational cubic spline interpolation.

2. Numerical comparison with the work of Fristch and Carlson [6] for monotonicity preserving also has been done. Furthermore the first derivative need not to be modified in which the monotonicity is not preserves. Meanwhile the shape preserving by using Fristch and Carlson [5] and Dougherty et al. [7] require the modification of the first derivative in which the shape violation is found. The rational cubic Ball interpolation do not required any additional new knots compare to the work ofLahtinen [8].

3. The sufficient condition for monotonicity of the rational cubic Ball interpolant is different from the sufficient condition for monotonicity of the rational cubic spline (cubic/cubic) of the work by Hussain and Sarfraz [13]. Thus our rational cubic Ball interpolant provide good alternative to the existing rational cubic spline scheme for monotonicity preserving interpolation.

The remainder of the paper is organized as follows. Section 2 give the review the rational cubic Ball interpolant with four parameters initiated Karim [2]. Section 3 discuss the methods to estimate the first derivatives values meanwhile Section 4 is devoted to the monotonicity preserving by using rational cubic Ball interpolant. The sufficient condition for the monotonicity of the rational cubic Ball interpolant will be derived. All numerical results are given in Section 5 including comparison with the work of Fristch and Carlson [6]. Section 6 gives the summary and conclusions to the paper.

2. Rational Cubic Ball Interpolant

In this section the rational cubic Ball with four parameters proposed by Karim [2] will be reviewed in details. Assuming that the scalar (or functional) data is given

i.e. $\{(x_i, f_i), i = 1,...,n\}$ where $x_1 < x_2 < ... < x_n$. Let $h_i = x_{i+1} - x_i$, $\Delta_i = (f_{i+1} - f_i)/h_i$ and $\theta = (x - x_i)/h_i$ with $0 \le \theta \le 1$.

For $x \in [x_i, x_{i+1}], i = 1, 2, ..., n-1,$

$$s(x) \equiv S_i(\theta) = \frac{P_i(\theta)}{Q_i(\theta)}, \tag{1}$$

Where

$$P_i(\theta) = \alpha_i f_i (1-\theta)^2 + A_{i1}(1-\theta)^2 \theta + A_{i2}(1-\theta)\theta^2 + \delta_i f_{i+1}\theta^2$$

and

$$Q_i(\theta) = \alpha_i (1-\theta)^2 + \beta_i (1-\theta)^2 \theta + \gamma_i (1-\theta)\theta^2 + \delta_i \theta^2.$$

The parameters $\alpha_i, \beta_i, \gamma_i, \delta_i > 0, i = 1, 2, ..., n-1.$ The rational cubic Ball interpolant in (1) satisfies the following C^1 conditions:

$$\begin{aligned} s(x_i) &= f_i, \quad s(x_{i+1}) = f_{i+1}, \\ s^{(1)}(x_i) &= d_i, \quad s^{(1)}(x_{i+1}) = d_{i+1}. \end{aligned} \tag{2}$$

Simple algebraic manipulation to (1) by using C^1 conditions in (2) will give the value of the unknown variable A_{ij}, $j = 1, 2.$ It is given as follows:

$$A_{i1} = \beta_i f_i + \alpha_i h_i d_i, A_{i2} = \gamma_i f_{i+1} - \delta_i h_i d_{i+1}. \tag{3}$$

Some observations can be made as follows:
When $\alpha_i = \delta_i = 1, \beta_i = \gamma_i = 2,$ the rational cubic Ball interpolant defined by Eq. (1) is reduce to the following standard cubic Ball polynomial in Hermite-like form (Karim [2]).

$$s(x) = f_i(1-\theta)^2 + (2f_i + h_i d_i)(1-\theta)^2\theta + (2f_{i+1} - h_i d_{i+1})(1-\theta)\theta^2 + f_i\theta^2. \tag{4}$$

Furthermore $S(x)$ can be rewritten as follows:

$$s(x) = (1-\theta) f_i + \theta f_{i+1} + \frac{h_i\theta(1-\theta)E_i(\theta)}{Q_i(\theta)}. \tag{5}$$

where

$$E_i(\theta) = \left[\alpha_i(d_i - \Delta_i)(1-\theta) + \delta_i(\Delta_i - d_{i+1})\theta + \theta(1-\theta)\Delta_i(\gamma_i - \beta_i)\right].$$

When $\beta_i, \gamma_i \to \infty,$ or $\alpha_i, \delta_i \to 0,$ rational interpolant in (5) converges to straight line given below:

$$s(x) = (1-\theta) f_i + \theta f_{i+1}. \tag{6}$$

Thus the rational cubic Ball interpolant of [2] has the capability to reproduce the straight line when the parameters satisfied: $\alpha_i, \delta_i \to 0,$ and $\beta_i, \gamma_i \to \infty.$

3. Determination of Derivatives

For monotonicity preserving interpolation the first derivative values can be estimated either by using arithmetic mean method (AMM) or geometric mean method (GMM). In this paper both methods will be used to estimate the values of the first derivatives, d_i. Below the mathematical formula of AMM:

At the end points x_1 and x_n

$$d_1 = \Delta_1 + (\Delta_1 - \Delta_2)\left(\frac{h_1}{h_1 + h_2}\right) \tag{7}$$

$$d_n = \Delta_{n-1} + (\Delta_{n-1} - \Delta_{n-2})\left(\frac{h_{n-1}}{h_{n-1} + h_{n-2}}\right) \tag{8}$$

At the interior points, $x_i, i = 2, ..., n-1$, the values of d_i are given as

$$d_i = \frac{h_{i-1}\Delta_i + h_i\Delta_{i-1}}{h_{i-1} + h_i}. \tag{9}$$

Meanwhile the GMM is defined as follows:
At the end points x_1 and x_n

$$d_1 = \begin{cases} 0, & \Delta_1 = 0 \quad \text{or} \quad \Delta_{3,1} = 0 \\ \Delta_1^{\left(1+\frac{h_1}{h_2}\right)} \Delta_{3,1}^{\left(-\frac{h_1}{h_2}\right)} & \text{otherwise} \end{cases} \tag{10}$$

$$d_n = \begin{cases} 0 & \Delta_{n-1} = 0 \text{ or } \Delta_{n,n-2} = 0 \\ \Delta_{n-1}^{\left(1+\frac{h_{n-1}}{h_{n-2}}\right)} \Delta_{n,n-2}^{\left(-\frac{h_{n-1}}{h_{n-2}}\right)} & \text{otherwise} \end{cases} \tag{11}$$

At interior points, $x_i, i = 2, ..., n-1$, the values of d_i are given as

$$d_i = \Delta_{i-1}^{\left(\frac{h_i}{h_{i-1}+h_i}\right)} \Delta_i^{\left(\frac{h_{i-1}}{h_{i-1}+h_i}\right)}. \tag{12}$$

with $\Delta_{3,1} = \dfrac{f_3 - f_1}{x_3 - x_1}$, and $\Delta_{n,n-2} = \dfrac{f_n - f_{n-2}}{x_n - x_{n-2}}$.

The AMM is simple and easy to used. Meanwhile the GMM will give the positive values for the first derivative, d_i if the data is monotone. Both methods have their own advantages.

4. Monotonicity-Preserving Using Rational Cubic Ball Interpolant

In this section the rational cubic Ball function defined in Section 2 will be used for monotonicity preserving for strictly monotone data sets. We begin with the following assumption:

Let $(x_i, f_i), i = 1, ..., n$ be a given monotone data set, where $x_1 < x_2 < ... < x_n$. For a monotonic increasing (decreasing), the necessary condition should be:

$$f_1 \le f_2 \le ... \le f_n \text{ (or } f_1 \ge f_2 \ge ... \ge f_n \text{ for monotonic decreasing)} \tag{13}$$

Equation (13) is equivalent with

$$\Delta_i \ge 0 \quad \text{(or } \Delta_i \le 0 \text{ for monotonic decreasing data)} \tag{14}$$

In this section, the necessary and sufficient condition for the C^1 monotonicity of rational cubic Ball interpolant will be derived in details. For monotonic (increasing or decreasing) the rational cubic Ball interpolant $s(x)$ in Eq. (1), the first derivative must satisfy:

$$d_i \ge 0, i = 1, 2, ..., n. \text{(for monotonic increasing)} \tag{15}$$

Now, $s(x)$ is monotonic increasing if and only if

$$s^{(1)}(x) \ge 0, \quad x_1 \le x \le x_n. \tag{16}$$

Now after some simplification the first derivative of the rational cubic Ball interpolant $s(x)$ is given by:

$$s^{(1)}(x) = \frac{\sum_{j=0}^{4} B_{ij} (1-\theta)^{4-j} \theta^j}{\left[Q_i(\theta)\right]^2}. \tag{17}$$

with

$B_{i0} = \alpha_i^2 d_i, B_{i1} = 2\alpha_i\left(\left(\delta_i + \gamma_i\right)\Delta_i - \delta_i d_{i+1}\right),$
$B_{i3} = 2\delta_i\left(\left(\beta_i + \alpha_i\right)\Delta_i - \alpha_i d_i\right), A_{i4} = \delta_i^2 d_{i+1}$ and
$B_{i2} = \alpha_i\left(\delta_i\left(4\Delta_i - d_i - d_{i+1}\right) + \gamma_i\left(\Delta_i - d_i\right)\right) + \beta_i\left(\gamma_i\Delta_i + \delta_i\Delta_i - \delta_i d_{i+1}\right).$

Furthermore, B_{i2} can be rewritten as follows:

$$B_{i2} = 4\alpha_i\delta_i\Delta_i + \beta_i\gamma_i\left(\Delta_i - \frac{\alpha_i d_i}{\beta_i} - \frac{\delta_i d_{i+1}}{\gamma_i}\right) + \beta_i\delta_i\left(\Delta_i - \frac{\alpha_i d_i}{\beta_i}\right) + \alpha_i\gamma_i\left(\Delta_i - \frac{\delta_i d_{i+1}}{\gamma_i}\right).$$

Now, $s^{(1)}\left(x\right) \geq 0$ if and only if $B_{ij} \geq 0, j = 0,1,2,3,4$. The necessary conditions for monotonicity are $d_i \geq 0, d_{i+1} \geq 0$ and the sufficient condition for monotonicity can be obtained from: $B_{ij} \geq 0, j = 0,1,2,3,4$. Clearly $B_{i0} \geq 0, B_{i4} \geq 0$. If the given data is strictly monotone (i.e. $\Delta_i > 0$), then from $B_{i1} \geq 0, B_{i2} \geq 0$ and $B_{i3} \geq 0$ will gives us the following conditions:

$B_{i1} \geq 0$ if

$$2\alpha_i\left(\left(\delta_i + \gamma_i\right)\Delta_i - \delta_i d_{i+1}\right) \geq 0. \tag{18}$$

$B_{i2} \geq 0$ if

$$4\alpha_i\delta_i\Delta_i + \beta_i\gamma_i\left(\Delta_i - \frac{\alpha_i d_i}{\beta_i} - \frac{\delta_i d_{i+1}}{\gamma_i}\right) + \beta_i\delta_i\left(\Delta_i - \frac{\alpha_i d_i}{\beta_i}\right) + \alpha_i\gamma_i\left(\Delta_i - \frac{\delta_i d_{i+1}}{\gamma_i}\right) \geq 0. \tag{19}$$

and

$B_{i3} \geq 0$ if

$$2\delta_i\left(\left(\beta_i + \alpha_i\right)\Delta_i - \alpha_i d_i\right) \geq 0. \tag{20}$$

Eq. (18), (19) and (20) provides the following inequalities:

$$\gamma_i \geq \frac{\delta_i d_{i+1}}{\Delta_i}. \tag{21}$$

$$\Delta_i - \frac{\alpha_i d_i}{\beta_i} - \frac{\delta_i d_{i+1}}{\gamma_i} \geq 0. \tag{22}$$

$$\beta_i \geq \frac{\alpha_i d_i}{\Delta_i}. \tag{23}$$

The sufficient conditions in (21), (22) and (23) can be stated as the following theorem.

Theorem 1: Given a strictly monotonic increasing set of data satisfying (13) or (14), there exist monotonic rational cubic Ball interpolating spline $s(x) \in C^1[x_1, x_n]$ involving free parameters α_i, δ_i that satisfy the following sufficient conditions:

$$\alpha_i, \delta_i > 0,$$

$$\gamma_i = \delta_i \left(\frac{d_i + d_{i+1}}{\Delta_i} \right), \beta_i = \alpha_i \left(\frac{d_i + d_{i+1}}{\Delta_i} \right). \tag{24}$$

Remark 1: The choices given in Eq. (24) satisfied the conditions in Eq. (18), (19) and (20). Thus the monotonicity of the rational cubic Ball interpolant is guaranteed.

Remark 2: If the data are constant on certain interval, i.e. $\Delta_i = 0$, then it is necessary to set $d_i = d_{i+1} = 0,$ hence $s(x) = f_i = f_{i+1}$ is a constant on the interval $[x_i, x_{i+1}], i = 1, 2, ..., n-1$. This shows that the rational interpolant is monotone.

5. Results and Discussion

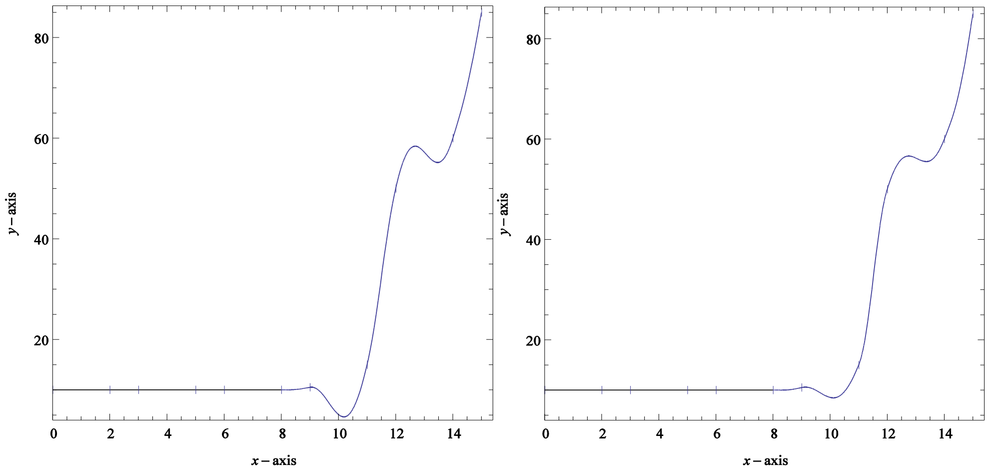
In order to illustrate the monotonicity preserving interpolation by using rational cubic Ball interpolation (cubic/cubic), two sets of monotone data taken from Akima [15] and Sarfraz et al. [9] were used. All the data sets are listed in Table 1 and Table 2 respectively.

Table 1. A monotone data from [15]

i	1	2	3	4	5	6	7	8	9	10	11
x_i	0	2	3	5	6	8	9	11	12	14	15
f_i	10	10	10	10	10	10	10.5	15	50	60	85
d_i (GMM)	0	0	0	0	0	0	1.35	14.021	18.297	14.620	36.596
d_i (AMM)	0	0	0	0	0	0	1.0833	24.0833	25	18.3333	31.6667

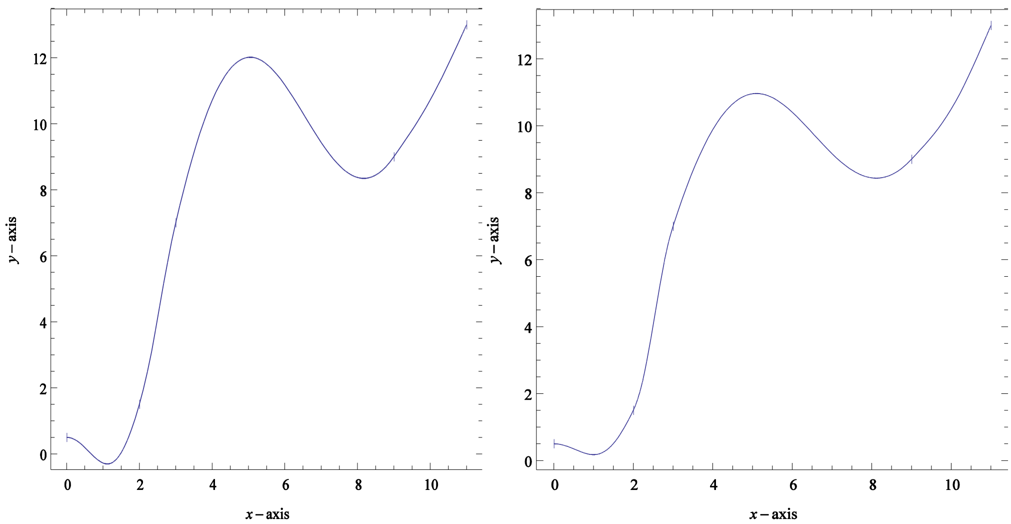
Table 2. A monotone data from [9]

i	1	2	3	4	5
x_i	0	2	3	9	11
f_i	0.5	1.5	7	9	13
d_i (GMM)	0.00266	2.4730	3.6850	1.2779	2.7734
d_i (AMM)	0 (-2.833)	3.8333	4.7619	1.5833	2.4167



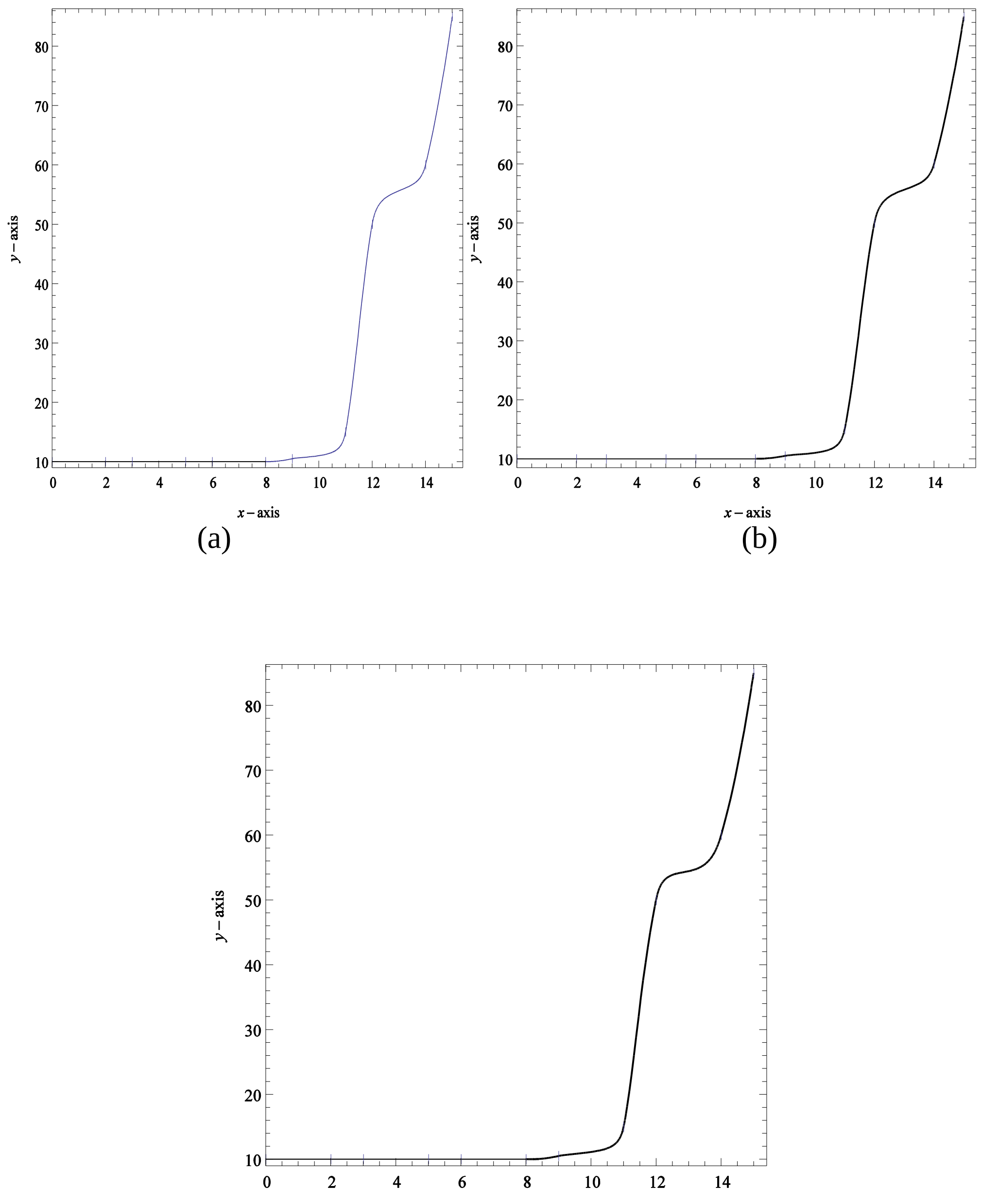
(a) (b)

Figure 1. Default Cubic Ball interpolation by using (a) AMM and (b) GMM for data in Table 1.



(a) (b)

Figure 2. Default Cubic Ball interpolation by using (a) AMM and (b) GMM for data in Table 2.



(a) (b)

(c)

Figure 3. Shape preserving by using AMM with (a) $\alpha_i = \delta_i = 1$ (b) $\alpha_i = \delta_i = 2$ and (c) $\alpha_i = 1, \delta_i = 2$ for data in Table 2.

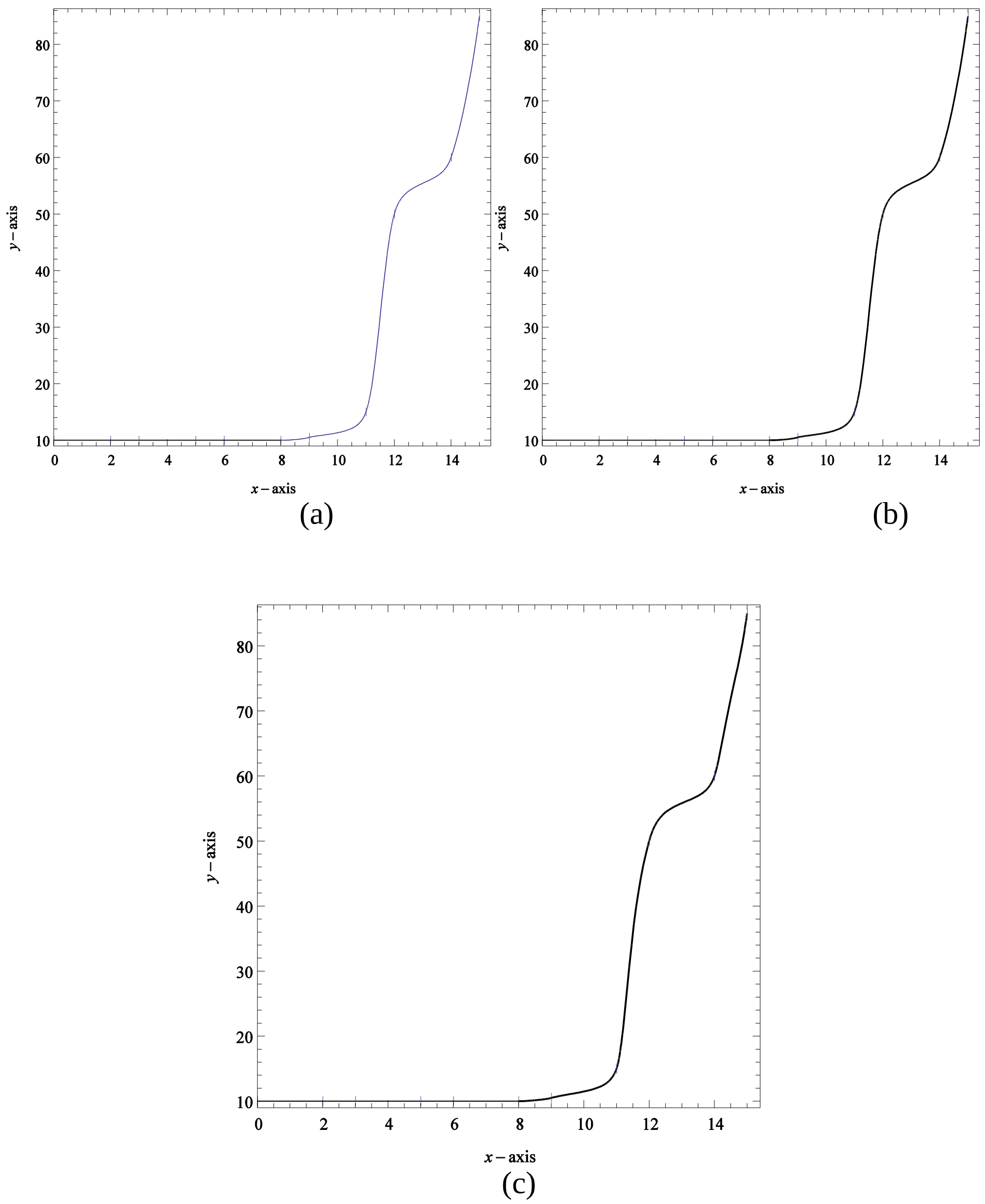
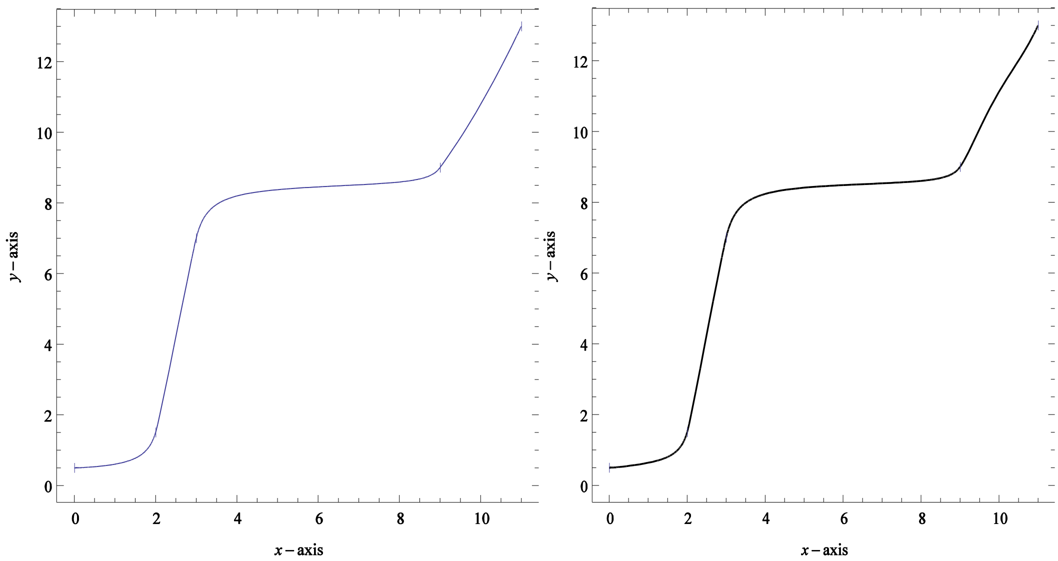
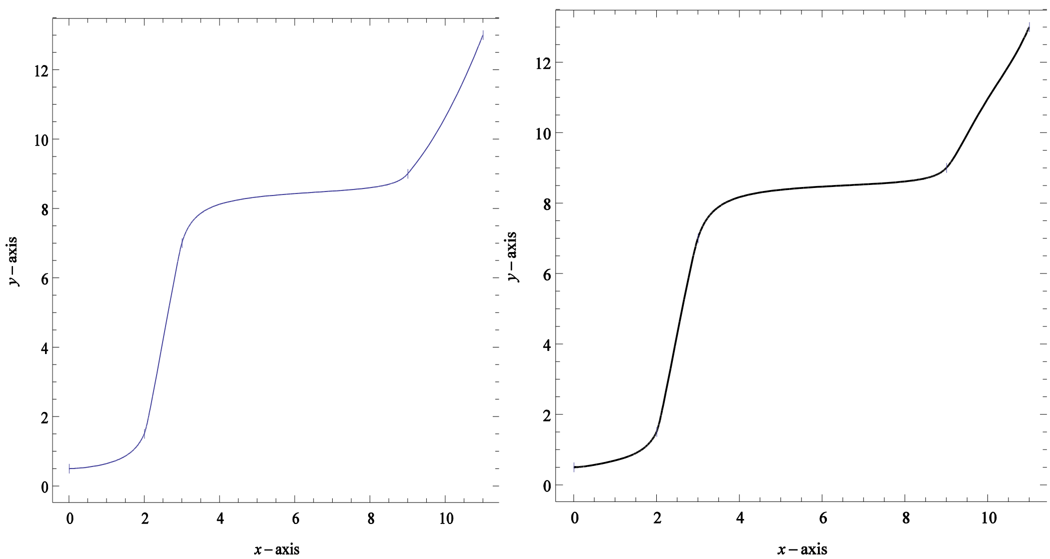


Figure 4. Shape preserving by using GMM with (a) $\alpha_i = \delta_i = 1$ (b) $\alpha_i = \delta_i = 2$ (c) $\alpha_i = 1, \delta_i = 2$ for data in Table 1.



(a) (b)

Figure 5. Shape preserving by using AMM with (a) $\alpha_i = \delta_i = 1$ (b) $\alpha_i = 1, \delta_i = 2$ for data in Table 2.



(a) (b)

Figure 6. Shape preserving by using GMM with (a) $\alpha_i = \delta_i = 1$ (b) $\alpha_i = 1, \delta_i = 2$ for data in Table 2.

Figure 1(a) and Figure 1(b) and Figure 2(a) and Figure 2(b) shows the default cubic Ball interpolation i.e. when $\alpha_i = \delta_i = 1, \beta_i = \gamma_i = 2,$ for data listed in Table 1 and Table 2 respectively. Figure 3(a), 3(b) and 3(c) and Figure 4(a), 4(b) and 4(c) shows the monotonicity preserving by using AMM and GMM for data sets listed in Table 1 respectively. Meanwhile Figure 5(a) and 5(b) and Figure 6(a) and 6(b) shows the monotonicity preserving by using AMM and GMM for data sets listed in Table 1 respectively.

From Figure 2 until Figure 6 it can be seen clearly that monotonicity preserving by using rational cubic Ball interpolant with four parameters give smooth and very visual pleasing interpolating curves. Furthermore the GMM method give more smooth monotonic interpolating curves as compare with the monotonic interpolating curve by using AMM method. In general, both methods are acceptable to estimate the first derivative values for monotonicity preserving by using rational cubic Ball interpolant. One question still remains to be answered: Which method give better results?

Table 3, Table 4, Table 5 and Table 6 gives the value for all shape parameters for Figure 3(a), Figure 4(a), Figure 5(b) and Figure 6(b) respectively. The other shape parameters can be calculated by using Eq. (24).

Finally Figure 7 and Figure 8 show the shape preserving interpolation by using Fristch and Carlson [6] methods for data in Table 1 and Table 2 respectively. Clearly the proposed rational cubic Ball interpolation with four parameters give better interpolating curves together with more smooth results compare to the work of Fristch and Carlson [6].

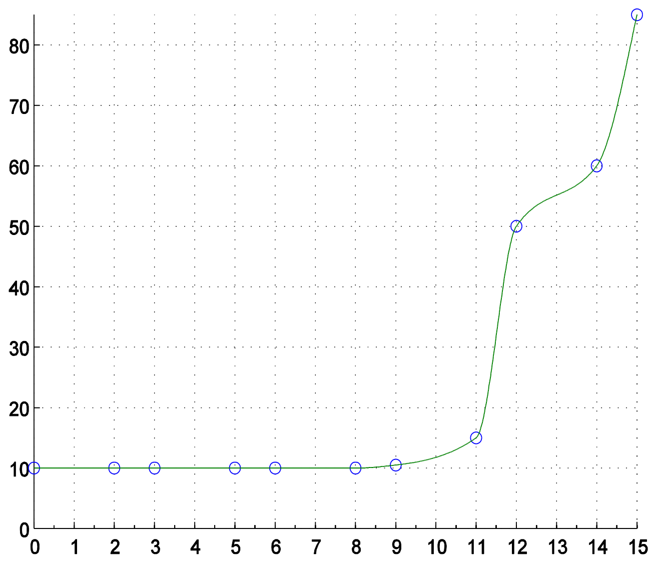


Figure 7. Shape preserving by using Fristch and Carlson [6] for data in Table 1.

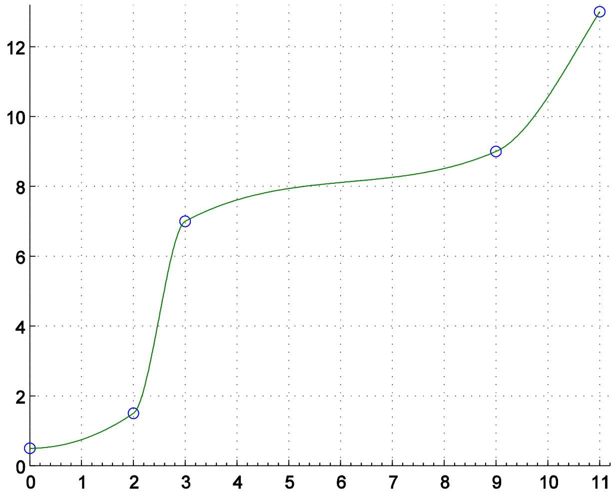


Figure 8. Shape preserving by using Fristch and Carlson [6] for data in Table 2.

Table 3. Shape parameters values for Figure 3(a)

i	1	2	3	4	5	6	7	8	9	10
α_i	-	-	-	-	-	1	1	1	1	1
β_i	-	-	-	-	-	2.17	11.19	1.40	8.67	2
γ_i	-	-	-	-	-	2.17	11.19	1.40	8.67	2
δ_i	-	-	-	-	-	1	1	1	1	1

Table 4. Shape parameters values for Figure 4(a)

i	1	2	3	4	5	6	7	8	9	10
α_i	-	-	-	-	-	1	1	1	1	1
β_i	-	-	-	-	-	2.7	8.83	0.92	6.58	2.05
γ_i	-	-	-	-	-	2.7	8.83	0.92	6.58	2.05
δ_i	-	-	-	-	-	1	1	1	1	1

Table 5. Shape parameters values for Figure 5(b)

i	1	2	3	4
α_i	1	1	1	1
β_i	7.67	10	19.04	2
γ_i	15.33	10	38.07	4
δ_i	2	2	2	2

Table 6. Shape parameters values for Figure 6(b)

i	1	2	3	4
α_i	1	1	1	1
β_i	4.95	10	14.89	2.03
γ_i	9.90	10	29.77	4.05
δ_i	2	2	2	2

Final Remark: For the Akima data sets, the monotonicity preserving is applied only on the interval [8, 15]. This is due to the fact that on the interval [0, 8] the rational cubic Ball interpolation will reproduce the straight line since on this interval since $\Delta_i = 0,\ i = 1,...,5.$

6. Conclusions

These paper discuss the use of rational cubic Ball interpolant with four parameters for monotonicity preserving. The sufficient conditions for the monotonicity of the rational cubic Ball interpolant are derived on two parameters i.e. β_i and γ_i. Meanwhile the remaining two parameters α_i and β_i are free parameters that can be further utilized to refine the final shape of the monotonic interpolating curves.

The first derivative is estimated by using AMM and GMM. Clearly the proposed rational cubic Ball interpolant give very smooth monotonic interpolating curves for both tested data sets. The free parameters α_i and β_i give extra degree of freedom to the user in controlling the final shape of the monotonic interpolating curves. The rational cubic Ball interpolant also can used to generate the C^2 monotonic interpolating curves. Works on bivariate interpolating also is underway by the author.

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