Curves of Constant Breadth in Galilean 3-Space

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Abstract

In this paper, we investigate the properties of curves of constant breadth in the three dimensional Galilean space and construct curves of constant breadth.

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1 Introduction

In 1778, Euler was introduced curves of constant breadth in plane [2]. After him, Fujiwara extended it to space curves. He had obtained a problem to determine whether there exists space curve of constant breadth or not, and defined breadth for space curves [4]. Furthermore, Blaschke defined the curve of constant breadth on the sphere [1]. Several results were obtained when the ambient space is the Euclidean space and the Lorentz-Minkowski space [5], [6] and [7].

In this paper, we study curves of constant breadth in the three dimensional Galilean space $G_3$ and construct it.

2 Preliminary Notes

The Galilean space $G_3$ is a Cayley-Klein space equipped with the projective metric of signature $(0,0,+,+)$. The absolute figure of the Galilean space
consist of an ordered triple \( \{ w, f, I \} \), where \( w \) is the ideal (absolute) plane, \( f \) is the line (absolute line) in \( w \) and \( I \) is the fixed elliptic involution of points of \( f \). In the non-homogeneous coordinates the similarity group \( H_8 \) has the form

\[
\begin{align*}
\bar{x} &= a_{11} + a_{12} x \\
\bar{y} &= a_{21} + a_{22} x + a_{23} y \cos \theta + a_{23} z \sin \theta \\
\bar{z} &= a_{31} + a_{32} x - a_{23} y \sin \theta + a_{23} z \cos \theta
\end{align*}
\]

(1)

where \( a_{ij} \) and \( \theta \) are real numbers. In what follows the real numbers \( a_{12} \) and \( a_{23} \) will play the special role. In particular, for \( a_{12} = a_{23} = 1 \), equation (1) defines the group \( B_8 \subset H_8 \) of isometries of the Galilean space \( G_3 \).

Let \( X = (x_1, y_1, z_1) \) and \( Y = (x_2, y_2, z_2) \) be vectors in \( G_3 \). A vector \( X \) is called isotropic if \( x_1 = 0 \), otherwise it is called non-isotropic. The Galilean scalar product of \( X \) and \( Y \) is defined by

\[
X \cdot Y = \begin{cases} 
    x_1 x_2, & \text{if } x_1 \neq 0 \text{ or } x_2 \neq 0, \\
    y_1 y_2 + z_1 z_2, & \text{if } x_1 = 0 \text{ and } x_2 = 0.
\end{cases}
\]

From this, the Galilean norm of a vector \( X \) in \( G_3 \) is given by \( ||X|| = \sqrt{X \cdot X} \) and all unit non-isotropic vectors are the form \((1, y_1, z_1)\).

The Galilean cross product of \( X \) and \( Y \) on \( G_3 \) is defined by

\[
X \times Y = \begin{vmatrix} 
    0 & e_2 & e_3 \\
    x_1 & y_1 & z_1 \\
    x_2 & y_2 & z_2 
\end{vmatrix},
\]

where \( e_2 = (0, 1, 0) \) and \( e_3 = (0, 0, 1) \).

Let \( \alpha : I \to G_3 \) be a curve in \( G_3 \) defined by \( \alpha(t) = (x(t), y(t), z(t)) \), where \( t \in I \subset \mathbb{R} \) and \( x(t), y(t), z(t) \) are smooth functions. A curve \( \alpha \) is called admissible if \( x'(t) \) is not equal to zero. For an admissible curve \( \alpha \), a unit speed curve \( \alpha \) is given by

\[
\alpha(s) = (s, y(s), z(s)),
\]

(2)

where \( s \) is a Galilean invariant of the arc-length of \( \alpha \). Then the curvature \( \kappa(s) \) and the torsion \( \tau(s) \) are given by

\[
\kappa(s) = \sqrt{y''(s)^2 + z''(s)^2}, \quad \tau(s) = \frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s))}{\kappa^2(s)},
\]

respectively.

On the other hand, the Frenet moving frame of an admissible curve \( \alpha \) is given by

\[
\begin{align*}
T(s) &= \alpha'(s) = (1, y'(s), z'(s)), \\
N(s) &= \frac{1}{\kappa(s)} \alpha''(s) = \frac{1}{\kappa(s)} (0, y''(s), z''(s)), \\
B(s) &= T(s) \times N(s) = \frac{1}{\kappa(s)} (0, -z''(s), y''(s)).
\end{align*}
\]

(3)
The vectors $T, N$ and $B$ are called the tangent vector, principal normal vector and binormal vector of $\alpha$, respectively. Also, the Frenet formulas may be written as (cf. [3])

\[
\begin{align*}
T'(s) &= \kappa(s) N(s), \\
N'(s) &= \tau(s) B(s), \\
B'(s) &= -\tau(s) N(s).
\end{align*}
\]

(4)

3 Space curves of constant breadth in $G_3$

In this section, we define space curves of constant breadth in the three-dimensional Galilean space $G_3$ and construct it.

Definition 3.1 A curve $\alpha : I \rightarrow G_3$ in the three-dimensional Galilean space $G_3$ is called a curve of constant breadth if there exists a curve $\beta : I \rightarrow G_3$ such that, at the corresponding points of curves, the parallel tangent vectors of $\alpha$ and $\beta$ at $\alpha(s)$ and $\beta(s^\ast)$ at $s, s^\ast \in I$ are opposite directions and the distance between these points is always constant. In this case, $(\alpha, \beta)$ is called a curve pair of constant breadth.

Let now $(\alpha, \beta)$ be a curve pair of constant breadth and $s, s^\ast$ be arc-length of $\alpha$ and $\beta$, respectively. Then we may write the following equation:

\[
\beta(s^\ast) = \alpha(s) + m_1(s) T(s) + m_2(s) N(s) + m_3(s) B(s),
\]

(5)

where $m_i (i = 1, 2, 3)$ are smooth functions of $s$. Differentiating (5) with respect to $s$ and using (3) we obtain

\[
\frac{d\beta}{ds} = \frac{d\beta}{ds^\ast} \frac{ds^\ast}{ds} = T^\ast \frac{ds^\ast}{ds} = (1 + m'_1 T + (m'_2 + m_1 \kappa - m_3 \tau) N + (m'_3 + m_2 \tau) B,
\]

(6)

where $T^\ast$ denotes the tangent vector of $\beta$. Since $T = -T^\ast$, from (6) we have

\[
\begin{align*}
1 + m'_1 &= -\frac{ds^\ast}{ds}, \\
m'_2 + m_1 \kappa - m_3 \tau &= 0, \\
m'_3 + m_2 \tau &= 0.
\end{align*}
\]

(7)

If $\varphi$ is the angle between the tangent vector of $\alpha$ and a given fixed direction, the curvature of $\alpha$ is $\kappa = \frac{dc}{ds}$. It follows that (7) can be rewritten as

\[
\begin{align*}
\frac{dm_1}{d\varphi} &= -f(\varphi), \\
\frac{dm_2}{d\varphi} &= -m_1 + m_3 \rho \tau, \\
\frac{dm_3}{d\varphi} &= -m_2 \rho \tau.
\end{align*}
\]

(8)

Here $f(\varphi) = \rho + \rho^\ast$ and $\rho = 1/\kappa, \rho^\ast = 1/\kappa^\ast$ are the radii of the curvatures of $\alpha$ and $\beta$, respectively.
Differentiating the second equation of (8) with respect to $\varphi$ and using (8) we obtain the following equation:

$$\rho \tau f = \rho \tau \frac{d^2 m_2}{d\varphi^2} + (\rho \tau)^3 m_2 - \frac{d}{d\varphi}(\rho \tau) (m_1 + \frac{dm_2}{d\varphi}).$$  \hspace{1cm} (9)

If the distance between the opposite points of $\alpha$ and $\beta$ is constant, then

$$||\alpha - \beta||^2 = m_1^2 + m_2^2 + m_3^2 = \text{constant},$$

which implies

$$m_1 \frac{dm_1}{d\varphi} + m_2 \frac{dm_2}{d\varphi} + m_3 \frac{dm_3}{d\varphi} = 0.$$  \hspace{1cm} (10)

Combining (8) and (10) we get

$$(m_2 + f) \left( \frac{dm_2}{d\varphi} - m_3 \rho \tau \right) = 0.$$  \hspace{1cm} (11)

First of all, let us suppose that $m_2 = -f$ on an open interval. Then (9) becomes

$$\rho \tau m_2 (1 + (\rho \tau)^2) + \rho \tau \frac{d^2 m_2}{d\varphi^2} - \frac{d}{d\varphi}(\rho \tau) \left( m_1 + \frac{dm_2}{d\varphi} \right) = 0.$$  \hspace{1cm} (12)

In the case of $m_1 = $non-zero constant, from the first equation of (8) we find $m_2 = 0$. Thus, from (12) we can show that $\rho \tau$ is constant, that is, $\tau/\kappa = \text{constant}$. Thus, we have the following theorem:

**Theorem 3.2** The space curve of constant breadth with the tangent component $m_1 = $non-zero constant and the principal normal component $m_2 = 0$ is a general helix in the three dimensional Galilean space.

Now, we assume that $\frac{dm_2}{d\varphi} = m_3 \rho \tau$. Then from the second equation of (8) we have $m_1 = 0$, it follows that $f = 0$. Thus (9) writes as

$$\frac{d^2 m_2}{d\varphi^2} - \frac{d}{d\varphi}(\rho \tau) \frac{dm_2}{d\varphi} + (\rho \tau)^2 m_2 = 0.$$  \hspace{1cm} (13)

To solve this equation, let

$$R(\varphi) = \frac{d}{d\varphi}(\rho \tau) \rho \tau, \quad S(\varphi) = (\rho \tau)^2$$  \hspace{1cm} (14)

and let the transformation be $z = \theta(\varphi)$ with $\frac{dz}{d\varphi} = (\pm S(\varphi)/a^2)^{1/2}$, where $a$ is any positive integer. Then, (13) may be written in the form

$$\frac{d^2 m_2}{dz^2} + \left( \frac{d^2 z}{dz^2} - R(\varphi) \frac{dz}{d\varphi} \right) \frac{dm_2}{dz} + S(\varphi) \left( \frac{dz}{d\varphi} \right)^2 m_2 = 0.$$  \hspace{1cm} (15)
Since \( \frac{d}{d\varphi} = (\pm S(\varphi)/a^2)^{1/2} \), equation (15) becomes

\[ \frac{d^2 m_2}{dz^2} + a^2 m_2 = 0, \]

and its general solution is given by

\[ m_2 = c \cos(az + b) = c \cos(\int_0^\varphi \rho \tau d\varphi + b), \]

which implies that from the second equation of (8) we have

\[ m_3 = -c \sin(\int_0^\varphi \rho \tau d\varphi + b), \]

where \( b \) and \( c \) are constants of integration.

Therefore, from \( m_1 = 0 \), (17) and (18) we can see that when the space curve \( \alpha \) is given, then an infinite number of \( \beta \) can be derived and the distance between the corresponding points of a pair curve of constant breadth is \( c \). On the other hand, the condition \( \frac{dm_2}{d\varphi} = m_3 \rho \tau \) is equal to \( m_2' = m_3 \tau \).

**Theorem 3.3** Let \( (\alpha, \beta) \) be a curve pair of constant breadth in \( G_3 \). If \( \alpha \) is a curve with \( m_1 = 0 \) and \( m_2' = m_3 \tau \), then a curve \( \beta \) is expressed as

\[ \beta = \alpha + c \cos(\int_0^\varphi \frac{\tau}{\kappa} d\varphi + b)N - c \sin(\int_0^\varphi \frac{\tau}{\kappa} d\varphi + b)B. \]

**Example 3.4** Consider a curve

\[ \alpha(s) = (s, -s \cos s + 2 \sin s, -s \sin s - 2 \cos s) \]

in \( G_3 \). Then the curve has \( \kappa = s \) and \( \tau = 1 \). Also, the Frenet frame of \( \alpha \) is given by

\[ T(s) = (1, s \sin s + \cos s, -s \cos s + \sin s), \]
\[ N(s) = (0, \cos s, \sin s), \]
\[ B(s) = (0, -\sin s, \cos s). \]

Since \( \kappa = \frac{d\varphi}{ds} \), in the case we have \( \varphi = \frac{1}{2}s^2 + d \). Thus, from (19), a curve \( \beta \) becomes

\[ \beta(s) = \left( s, -s \cos s + 2 \sin s + \cos s \cos(\frac{1}{2}s) - \sin s \sin(\frac{1}{2}s), \right. \]
\[ \left. -s \sin s - 2 \cos s + s \cos(\frac{1}{2}s) + \sin s \sin(\frac{1}{2}s) \right) \]

with \( b = 0, c = 1 \) and \( d = 0 \).

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Fig. 1. A curve $\alpha(s)$ in Example 3.4.

Fig. 2. A curve $\beta(s)$ in Example 3.4.

References


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