Pattern Search for Portfolio Selection

Joseph Ackora-Prah
Department of Mathematics
Kwame Nkrumah University of Science and Technology, Kumasi, Ghana

Samuel Asante Gyamerah
Department of Mathematics
Kwame Nkrumah University of Science and Technology, Kumasi, Ghana

Perpetual Saah Andam
Department of Mathematics
Kwame Nkrumah University of Science and Technology, Kumasi, Ghana

Daniel Gyamfi
Department of Mathematics
Kwame Nkrumah University of Science and Technology, Kumasi, Ghana

Copyright © 2014 Joseph Ackora-Prah et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, we present a real life application of Pattern Search (PS) as a heuristic algorithm for the selection of an optimal portfolio of the Ghana Stock Exchange (GSE) market. Practical constraint (boundary and cardinality) model was formulated and a Mesh Adaptive Direct Search method was applied to solve the model. From the computational results, we show that PS is a powerful and efficient optimization tool when an investor wants to select an optimal portfolio and allocate weights to the portfolio.

Keywords: Portfolio Selection, Pattern Search, Mesh Adaptive Direct Search
1 Introduction

Portfolio Selection represents a problem where investors and portfolio managers search for the best firms of the market. Variance (Risk) and Expected return are the most important parameters with regard to Portfolio Selection Problems (PSP). One of the major contributions to the PSP is the Markowitz mean-variance model which states that for any specific return rate, the minimum investment risk can be derived by minimizing the variance of the Portfolio; or for any given risk level which the investor can tolerate, the maximum returns can be derived by maximizing the expected returns of a Portfolio [9].

Even though the Markowitz mean-variance model is the fundamental theory of modern portfolio theory (MPT), direct application of this model is not pragmatic mainly due to the fact that it is simplified with some unrealistic assumptions. It assumes an ideal market without taxes, administration cost or transaction costs. In reality however, investors usually face restrictions such as cardinality and bounding constraints. The addition of more relevant constraints makes the model more difficult to solve. When Markowitz mean-variance model is extended to include these constraints, the model is transformed from quadratic optimization model to quadratic mixed-integer problem (QMIP) which is NP-hard [14].

Many researchers have investigated a variety of approach to solve the constrained portfolio selection problem using different exact techniques [6][7]. However, these exact techniques are sometimes unsuccessful in finding an optimal solution in a realistic time interval and are computationally ineffective when applied to large-scale problems. Since QMIPs are hard to solve, different heuristic methods have been used to solve the constrained PSP in the mean-variance context. Fernández and Gómez applied a Hopfield Neural Network Heuristic to the PSP with cardinality and bounding constraints [1]. Koshino et al. proposed an improved method of Particle Swarm Optimization (PSO), which is the inertia weights approach (IWA) and the constriction factor approach (CFA) to the PSP [12]. Schaerf applied a hill climbing algorithm which is a local search method to solve the constrained PSP [4]. Several works have applied GA to solve the PSP with various constraints [15][3]. Ackora-Prah et al. applied three types of GA crossovers in the selection of a portfolio with boundary and cardinality constraints [10] and Chang et al. presented three heuristic algorithms based upon genetic algorithm, tabu search and simulated annealing for finding the cardinality constrained efficient frontier [15].

In this paper, we show that portfolio selection problems containing cardinality and boundary constraints can be successfully solved by Pattern Search (PS). The attraction of this approach is that it is effectively independent of the ob-
jective function adopted and can be solved even when the objective function is highly non-linear and discontinuous.

2 Pattern Search (PS) as a Heuristic Algorithm

Heuristics are methods designed for solving problems when conventional methods are not able to find solutions. That is, classical optimization techniques have limited scope in practical applications. Although Heuristic may produce results, they may also be used with optimization algorithms to revamp their efficiency.

In 1961, Robert Hooke and T.A. Jeeves coined the phrase “direct search” [13]. They used “direct search” to describe a sequential examination of trial solutions involving comparison of each trial solution with the “best” obtained up to that time together with a strategy for determining (as a function of earlier results) what the next trial solution will be.

Pattern search method is a type of direct search method for which the rules of generating the trial points follow precise calculations and for which convergence for stationary points can be obtained from arbitrary starting points. It can be used to minimize a function $S(\vartheta)$ of several variables $\vartheta = (\vartheta_1, \vartheta_2, \vartheta_3, \cdots, \vartheta_x)$. The argument $\vartheta$ is varied until the minimum of $S(\vartheta)$ is obtained. The method does not require the gradient of the problem to be optimized. Pattern Search can therefore be used on functions that are not continuous or differentiable. Such optimization methods are also known as direct-search, derivative-free, or black-box methods.

The pattern search algorithm generates a set of search points to approach an optimal point. Around each search point, an area, called a mesh, is formed by adding the current point to a scalar multiple of a set of vectors called a pattern. If the search finds a point in the mesh that improves the objective function at the current point, the new point becomes the current point at the next step of the algorithm.

The generalized pattern search (GPS), mesh adaptive search (MADS), generating set search (GSS) are types of PS. GPS algorithm uses fixed direction vectors whilst MADS algorithm uses random vectors to define a mesh. The GSS algorithm is identical to the GPS algorithm, except when there is the presence of linear constraints, and when the current point is near a linear constraint boundary.
2.1 Mesh Adaptive Direct Search (MADS)

The MADS algorithm is a directional direct search method that iterate until the optimal point is found. During the $k_{th}$ iteration of the MADS, each trial point has on the mesh

$$M_k = \{ x + \Delta^m_k D_z : x \in V_k, z \in \mathbb{N}^{n_D} \} \subset \mathbb{R}^n$$

where $V_k \subset \mathbb{R}^n$ is the set of all points evaluated by the start of the iteration, $D$ is a fixed matrix in $\mathbb{R}^{n \times n_D}$ consisting of $n_D$ columns depicting directions, and $\Delta^m_k \in \mathbb{R}_+$ is the mesh size parameter.

Each iteration in the algorithm is divided into two major steps called Search and Poll. A Search is an algorithm that runs before a Poll. The Search tries to find an optimal point (point with lower objective function value) than the current point. If the Search is successful in finding the optimal point, this point becomes the current point and no Polling is done at the iteration. Failure to find an optimal point cause the Pattern Search to perform a Poll. The Search method helps to maintain a better local solution and obtain a global solution.

The Poll step guarantees theoretical convergence. It builds a set of candidates, $P_k$ called the Poll Set, defined as:

$$P_k = \{ x_k + \Delta^m_k d : d \in \Delta_k \} \subseteq M_k$$

where $\Delta_k$ is the set of Polling direction constructed by taking combinations of the set of directions $D$. The chance of $D_k$ depends on the MADS implementation. The Poll size parameter $\Delta^p_m$ explains the maximum distance at which the Poll trial points are generated from the current iterate $x_k$, which is also called the Poll center. The independent evaluation of the mesh and poll size parameters is modeled to ensure the set of poll directions become dense in the unit sphere—to ensure that every direction is explored.

A final update step is performed after the search and the poll step. Initially, It finds out if the iteration is a success or a failure and it then updates the mesh and poll sizes. If the iteration is a failure, these sizes are probably decreased, and increased otherwise. The decrease can not be of the same rate for the two parameters: the mesh size is reduced faster than the poll size, which allows additional possible directions as the mesh becomes thinner.

The most attractive feature of the algorithm is that it can converge globally to a solution fulfilling local optimality conditions.
2.1.1 Algorithm for MADS

The algorithm for MADS

1. Initialization
   (a) Starting Point: \( x_o \in \chi \)
   (b) Initial mesh and poll size parameters: \( \triangle_m^0, \triangle_p^0 > 0 \)
   (c) \( k \leftarrow 0 \)

2. Iterate \( k \)
   (a) SEARCH (optional)
      i. Evaluate \( f \) and \( h \) on a finite set \( S_k \subset M_k \)
   (b) POLL (optional if SEARCH was successful)
      i. Compute poll directions and trial points set \( P_k \)
      ii. Evaluate \( f \) and \( h \) on \( P_k \)

3. Updates
   (a) Determine success or failure of iteration \( k \)
   (b) Update incumbent \( x_{k+1} \)
   (c) Update mesh and poll size parameters
   (d) \( k \leftarrow k+1 \)
   (e) goto 1 if no stopping condition is met

where \( f \) is the objective function, \( h \) is the constraint function and \( \chi \) contains bound constraints

3 Portfolio Selection Model

We note that in a two-parameter model, the returns on any portfolio are normally distributed and can be described from the portfolio mean and standard deviation. Hence, with a normal return distribution, an investor who is risk-averse only considers a portfolio which has the largest possible expected return at a given risk rate, and the smallest possible risk rate at a given expected return. Portfolio with these two characteristics is considered to be efficient, and the collection of portfolios with the two properties is called efficient set. The portfolio properties can be summarized into a single one which states that for a portfolio to be efficient there must not be another portfolio with the same or higher expected return that has lower standard deviation. Consequently an efficient portfolio must:
1. have the maximum possible expected return given the variance, and
2. have the smallest possible variance given its expected return.

A portfolio satisfying condition (2) is called a Minimum Variance Portfolio and it is considered to be the solution of the following optimization problem:

\[
\text{Min } \sigma^2(R_p) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ip}x_{jp}\sigma_{ij} \tag{1}
\]

with the following constraints:

\[
\sum_{i=1}^{n} x_{ip}E(R_i) = E(R_e) \tag{2}
\]
\[
\sum_{i=1}^{n} x_{ip} = 1 \tag{3}
\]
\[
x_{ip} \geq 0, i = 1, \ldots, n \tag{4}
\]

where \( x_{ip} \) is the proportion of portfolio funds invested in security \( i \) in portfolio \( p \), \( n \) is the number of securities available, \( R_i \) is the returns on security \( i \), \( \sigma_{ij} = \text{cov}(R_i, R_j) \) represents the covariance between returns on securities \( i \) and \( j \) \( E(R_i) \) is the expected return on security \( i \).

Equation 1 is the objective function that minimizes the total variance (risk) associated with the portfolio, Equation 2 ensures that the portfolio has a specific expected return of \( E(R_e) \), Equation 3 defines the budget constraint (all the money available should be invested) for a feasible portfolio and Equation 4 ensures that the weights are nonnegative, that is no short sales are allowed. The set of efficient portfolios can be traced out by solving the model Equation 1–4 repeatedly with different value of expected return at each time.

By introducing a risk aversion parameter \( w \in [0,1] \), the sensitivity of the investor to the risk can be defined in the model as follows:

\[
\text{Minimize } w \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ip}x_{jp}\sigma_{ij} \right] - (1 - w) \left[ \sum_{i=1}^{n} x_{ip}E(R_i) \right] \tag{5}
\]

Subject to

\[
\sum_{i=1}^{n} x_{ip} = 1
\]
\[
\sum \varepsilon_i z_i = k
\]
\[
\varepsilon_i z_i \leq x_{ip} \leq \sigma_i z_i, \quad i = 1, 2, 3, \ldots, n.
\]
where the constraints above are the budget constraint, cardinality and boundary (floor and ceiling) constraints respectively. Budget constraints ensures that all the money available to the investor be invested in the portfolio. The cardinality constraint imposes a limit on the number of assets in the portfolio either to reduce transaction costs or to simplify the management of the portfolio. The bounding constraint limit the ratio of each asset in the portfolio to lie between the lower and upper bounds so as to avoid very small (or large) and unrealistic holdings.

The Cardinality Constrained Mean Variance (CCMV) model above is a mixed integer quadratic programming problem for which there exists no efficient algorithm. It may be seen as two subproblems: the selection of assets and the determination of the proportions of the selected assets.

4 The Data

The Ghana Stock Exchange (GSE) has 37 listed companies. Among the functions of the GSE is to provide facilities and framework to the public for the purchase and sales of bonds, stocks, shares and other securities. In this research, we worked with a pool of 6 stocks and the data was from 2007 – 2013. We assumed that:

1. All investors are risk averse.
2. Short selling is not allowed.
3. Dividends are not included.
4. closing prices are taken at the end of each day.

<table>
<thead>
<tr>
<th>Year</th>
<th>Stock 1</th>
<th>Stock 2</th>
<th>Stock 3</th>
<th>Stock 4</th>
<th>Stock 5</th>
<th>Stock 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2013</td>
<td>0.001453133</td>
<td>0.001641081</td>
<td>0.001091249</td>
<td>0.001465816</td>
<td>0.001508429</td>
<td>0.001096136</td>
</tr>
<tr>
<td>2012</td>
<td>0.000429243</td>
<td>0.000543547</td>
<td>0.00071918</td>
<td>0.000225605</td>
<td>0.000957417</td>
<td>-0.000109301</td>
</tr>
<tr>
<td>2011</td>
<td>0.000745183</td>
<td>-6.17E-05</td>
<td>-0.002628853</td>
<td>-0.000658221</td>
<td>-3.41E-05</td>
<td>0.000107973</td>
</tr>
<tr>
<td>2010</td>
<td>0.000752527</td>
<td>0.000689607</td>
<td>0.00119565</td>
<td>0.002395387</td>
<td>0.000268337</td>
<td>0.000128048</td>
</tr>
<tr>
<td>2009</td>
<td>-0.001339323</td>
<td>-0.002004711</td>
<td>0.000382691</td>
<td>-0.000723365</td>
<td>-0.000717211</td>
<td>-0.000865775</td>
</tr>
<tr>
<td>2008</td>
<td>0.001244192</td>
<td>0.000533048</td>
<td>0.001103673</td>
<td>0.000174978</td>
<td>0.000847891</td>
<td>0.001414388</td>
</tr>
<tr>
<td>2007</td>
<td>-0.000363783</td>
<td>0.0016855</td>
<td>0.000678282</td>
<td>0.001297666</td>
<td>0.000871102</td>
<td>0.001123366</td>
</tr>
</tbody>
</table>

The covariance matrix and mean return for each asset portfolio are given in the tables below:
Table 2: The Covariance matrix

<table>
<thead>
<tr>
<th></th>
<th>Stock 1</th>
<th>Stock 2</th>
<th>Stock 3</th>
<th>Stock 4</th>
<th>Stock 5</th>
<th>Stock 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>STOCK 1</td>
<td>9.45E-07</td>
<td>7.61E-07</td>
<td>6.47E-08</td>
<td>4.18E-07</td>
<td>4.89E-07</td>
<td>5.04E-07</td>
</tr>
<tr>
<td>STOCK 2</td>
<td>7.61E-07</td>
<td>1.55E-06</td>
<td>5.06E-07</td>
<td>1.02E-06</td>
<td>8.23E-07</td>
<td>8.25E-07</td>
</tr>
<tr>
<td>STOCK 3</td>
<td>6.47E-08</td>
<td>5.06E-07</td>
<td>1.83E-06</td>
<td>9.39E-07</td>
<td>4.42E-07</td>
<td>3.21E-07</td>
</tr>
<tr>
<td>STOCK 4</td>
<td>4.18E-07</td>
<td>1.02E-06</td>
<td>9.39E-07</td>
<td>1.35E-06</td>
<td>4.47E-07</td>
<td>4.04E-07</td>
</tr>
<tr>
<td>STOCK 5</td>
<td>4.89E-07</td>
<td>8.23E-07</td>
<td>4.42E-07</td>
<td>4.47E-07</td>
<td>5.50E-07</td>
<td>4.75E-07</td>
</tr>
<tr>
<td>STOCK 6</td>
<td>5.04E-07</td>
<td>8.25E-07</td>
<td>3.21E-07</td>
<td>4.04E-07</td>
<td>4.75E-07</td>
<td>6.76E-07</td>
</tr>
</tbody>
</table>

Table 3: The mean returns, $r_i$ for each asset

<table>
<thead>
<tr>
<th></th>
<th>Stock 1</th>
<th>Stock 2</th>
<th>Stock 3</th>
<th>Stock 4</th>
<th>Stock 5</th>
<th>Stock 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean ret.</td>
<td>0.00041731</td>
<td>0.000432339</td>
<td>0.000363125</td>
<td>0.000596838</td>
<td>0.000413548</td>
<td>0.000528838</td>
</tr>
</tbody>
</table>

5 Discussion

We use the constituent stock of the 6 largest market capitalizations listed in the Ghana Stock Exchange (GSE) as the investment universe. The yearly financial stock returns used for this research are retrieved from the GSE database for the period of time from 2007 to 2013. Complete Poll and Search was used in the PS implementation. MADS search method which uses random vectors to define a mesh was applied. To further illustrate the performance difference of Pattern Search heuristics, a model (objective fitness function) was formulated to solve for the optimal portfolio.

5.1 Results

All simulations in this work were executed using MATLAB. Illustrated below are the results obtained via the PS:

<table>
<thead>
<tr>
<th>Objective function value</th>
<th>Average Return of Portfolio</th>
<th>Variance of Portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.349117811236347e-04</td>
<td>4.704877262804416e-04</td>
<td>6.641640331722724e-07</td>
</tr>
</tbody>
</table>
Figure 1: MADS as applied to portfolio selection

<table>
<thead>
<tr>
<th>Weight, $x_1$</th>
<th>0.1534808476766595</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight, $x_2$</td>
<td>0.158674796422322</td>
</tr>
<tr>
<td>Weight, $x_3$</td>
<td>0.13654952281697</td>
</tr>
<tr>
<td>Weight, $x_4$</td>
<td>0.201481898625692</td>
</tr>
<tr>
<td>Weight, $x_5$</td>
<td>0.198639710744222</td>
</tr>
<tr>
<td>Weight, $x_6$</td>
<td>0.151173194249471</td>
</tr>
</tbody>
</table>

6 Conclusion

Computational results are presented for the Pattern Search algorithm in finding solution to the portfolio selection problem. A return of 4.704877262804416e-04 and an associated risk of 6.641640331722724e-07. Any investor wishing to get such a return and its associated risk rate should allocate 15.3481% of his budget on stock 1, 15.8674% on stock 2, 13.6550% on stock 3, 20.1482% on stock 4, 19.8640% on stock 5 and 15.1173% on stock 6.

The results we obtained showed that Pattern Search method is a tool for solving portfolio selection problems with cardinality and boundary constraints.
Acknowledgements. We express our gratitude to the Almighty God and the Department of Mathematics, Kwame Nkrumah University of Science and Technology for providing us resources to help complete this research successfully.

References


Received: June 1, 2014