Applied Mathematical Sciences, Vol. 8, 2014, no. 139, 6927 - 6935
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2014.48678

On the Method of Lyapunov Constant-Sign Functionals

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Abstract

In the work there is investigated the stability of the zero solution of a non-autonomous functional differential equation of the delayed type by means of limiting equations and Lyapunov constant-sign functional. An appropriate illustrating example is given.

Keywords: Lyapunov constant-sign functional, limit equations, stability problem

1. Introduction. Basic definitions and limiting equations

Suppose R is a real axis, R^n is a real linear space of n-vectors x with a norm |x|, $h = const > 0$ is a real number, C is the Banach space of continuous

functions $\varphi:[-h,0]\to R^n$ with a norm $\|\varphi\|=\sup(|\varphi(s)|,-h\le s\le 0)$, C_H is a space $\{\varphi\in C:\|\varphi\|<H, H>0\}$. For a continuous function $x:]-\infty,+\infty[\to R^n$ and every $t\in R$, the function $x_t(s)\in C_H$ is defined by the equality $x_t(s)=x(t+s), s\in[-h,0]$. A right-hand derivative is denoted by $\dot{x}(t)$.

The functional differential equation with a finite delay

$$\dot{x}(t)=f(t,x_t), f(t,0)\equiv 0 \tag{1}$$

is considered, where $f:R^+\times C_H\to R^n$ is a continuous function which satisfies the following assumptions [1, 4].

Assumption 1. The function $f=f(t,\varphi)$ is bounded in each set $R^+\times\overline{C}_r, \overline{C}_r=\{\varphi\in C:\|\varphi\|\le r<H\}$:

$$|f(t,\varphi)|\le m(r),(t,\varphi)\in R^+\times\overline{C}_r.$$

The smoothing of the solutions of Eq. (1) as t increases follows from this assumption and, in particular, if $x=x(t,\alpha,\varphi),(\alpha,\varphi)\in R^+\times C_H$ is a solution of Eq.(1) which satisfies the initial condition $x_\alpha=\varphi$, then, for the values $t\ge\alpha+h$, the function $x_t\in\Gamma$, where $\Gamma\subset C_H$ is the union of the family of imbedded compact sets $\Gamma=\bigcup_{n=1}^{\infty}K_n$, $K_1\subset K_2\subset\ldots\subset K_n\subset\ldots$ [1].

Assumption 2. The function $f=f(t,\varphi)$ satisfies the Lipschitz condition with respect to φ in each compact set $K\subset C_H$, that is for any $t\in R^+$ and $\varphi_1,\varphi_2\in K$ the inequality holds:

$$|f(t,\varphi_1)-f(t,\varphi_2)|\le l(K)\|\varphi_1-\varphi_2\|.$$

The uniqueness of the solution $x=x(t,\alpha,\varphi),(\alpha,\varphi)\in R^+\times C_H$ of Eq.(1) follows from the second assumption [3].

Assumption 3. The function $f=f(t,\varphi)$ is uniformly continuous with respect to $(t,\varphi)\in R^+\times K$ for each compact set $K\subset C_H$.

Under this assumption, 3 the family of shifts $\{f^\tau(t,\varphi)=f(t+\tau,\varphi),\tau\in R^+\}$ is precompact in a certain space F of continuous functions $f:R^+\times\Gamma\to R^n$ [1].

A function $f^*: R^+ \times \Gamma \to R^n$ is called as the limiting function to $f(t,\varphi)$, if a sequence $t_n \to +\infty$ exists such that $f\ (t+t_n,\varphi)$ is convergent to $f^*(t,\varphi)$ in F.

The equation

$$\dot{x}(t) = f^*(t, x_t) \tag{2}$$

is called the limiting one to (1). The domain of the limiting equation is defined by $R \times \Gamma$. The uniqueness of solution of Eq. (2) follows from the second assumption.

The correlation between the solutions of Eq.(1) and Eq.(2) is determined by the following theorem [1].

Theorem 1. Suppose the function $f^*: R^+ \times \Gamma \to R^n$ is a limiting function to f in F with respect to a sequence $t_n \to +\infty$, and sequences $\{\alpha_n \in R^+\}$ and $\{\varphi_n \in \Gamma\}$ are such, that $\alpha_n \to \alpha \in R^+$, $\varphi_n \to \varphi \in \Gamma$ at $n \to \infty$. Then, if $x = x(t, t_n + \alpha_n, \varphi_n)$ are solutions of Eq.(1), and $x^* = x^*(t,\alpha,\varphi)$ is a solution of limiting Eq.(2), defined for $t \in [\alpha - h, \beta[$, the sequence of functions $x = x(t+t_n, t_n + \alpha_n, \varphi_n)$ converges to $x^* = x^*(t,\alpha,\varphi)$ uniformly with respect to $t \in [\alpha - h, \gamma[$ for every $\gamma < \beta$.

2. Basic results. Stability theorems

We will investigate the problem of the stability on the base of Lyapunov constant-sign functionals. We shall use the following definitions.

Definition 1. The solution $x = 0$ of Eq.(1) is stable with respect to set $\Lambda \subset C_H$, if, for any $\varepsilon > 0$ one can get $\delta = \delta(\varepsilon) > 0$, so that for $\varphi \in \Lambda \cap \{\|\varphi\| < \delta\}$ it is true that $|x_t(t,0,\varphi)| < \varepsilon$ for each solution $x(t,0,\varphi)$ of Eq.(1) for any $t \geq 0$.

Definition 2. The solution $x = 0$ of Eq.(1) is uniformly asymptotically stable with respect to set $\Lambda \subset C_H$, if it is stable with respect to $\Lambda \subset C_H$ and a $\Delta > 0$ exists, so that for any $\varepsilon > 0$ one can get $T = T(\varepsilon) > 0$, so that for every $\varphi \in \Lambda \cap \{\|\varphi\| < \Delta\}$ it is true that $\|x_t(0,\varphi)\| < \varepsilon$ for any $t \geq T$.

Definition 3. The solution $x = 0$ is a point of uniform attraction for the whole family of limiting equations $\{\dot{x}(t) = f^*(t, x_t)\}$ with respect to set $\Lambda \subset C_H$, if a Δ exists, so that for any $\varepsilon > 0$ there is $T = T(\varepsilon) > 0$, so that

for any solution $x^*(t,0,\varphi)$, $\varphi \in \Lambda \cap \{\|\varphi\| < \Delta\}$ of any equation $\dot{x}(t) = f^*(t, x_t)$ for any $t \geq T$ the inequality $\|x_t^*(0,\varphi)\| < \varepsilon$ holds.

Suppose $V : R^+ \times C_H \to R^+$ is a certain continuous functional, $x = x(t,\alpha,\varphi), (\alpha,\varphi) \in R^+ \times C_H$ is a certain solution of Eq.(1). Along this solution the functional V is a continuous time-dependent function $V(t) = V(t, x_t(\alpha,\varphi))$. For this function it is possible to define an upper right-hand derivative $\dot{V}(t,\varphi)$.

Let us denote as $\omega_i(u)$ continuous strictly monotonically increasing functions $\omega_i : R^+ \to R^+, \omega_i(0) = 0$.

Definition 4. Let us define a set for the functional $V(t,\varphi)$:

$$V^{-1}(\infty,0) = \{\varphi \in C_H : \exists \varphi_n \in C_H, \exists t_n \to +\infty : \varphi_n \to \varphi, V(t_n,\varphi_n) \to 0, n \to +\infty\}.$$

The definitions which have been introduced enable us to derive the sufficient conditions of stability and asymptotic stability when a non-negative functional with a non-positive derivative exists.

Let us prove the following theorem about stability of zero solution $x = 0$ of Eq.(1) using a constant-sign functional V.

Theorem 2. Suppose that:

1) a continuous functional $V : R^+ \times C_H \to R^+$ exists, so that $V(t,\varphi) \geq 0, V(t,0) \equiv 0, \dot{V}(t,\varphi) \leq 0, (t,\varphi) \in R^+ \times C_H$;

2) the solution $x = 0$ is a point of uniform attraction for solutions $\{\dot{x}(t) = f^*(t, x_t)\}$ with respect to the set $\Lambda_0 = V^{-1}(\infty,0)$.

Then the solution $x = 0$ is stable by Lyapunov.

Proof.

We will assume that the position of equilibrium $x = 0$ of Eq. (1) is unstable. Then, at a certain $\varepsilon_0 : 0 < \varepsilon_0 < H$ there is a moment $\alpha > 0$, a sequence $\{\varphi_n : \|\varphi_n\| \to 0, n \to \infty\}$, so that for the solutions of Eq.(1) $x^n(t) = x(t,\alpha,\varphi_n)$ the following is true:

$$\|x_{t_n}^n(\alpha,\varphi_n)\| = \varepsilon_0 \tag{3}$$

at a certain $t = t_n$. From the uniqueness of $x = 0$, $t_n \to +\infty$ follows.

Suppose a $\Delta > 0$ is a number, defined by condition 2 of the theorem. We assume $\delta_0 = \frac{1}{2}\min(\varepsilon_0, \Delta)$. Then for the solutions $x^n(t) = x(t,\alpha,\varphi_n)$ there is a

sequence $t_n^{\delta_0} \le t_n$, so that:

$$\left\| x_{t_n^{\delta_0}}(\alpha,\varphi_n) \right\| = \delta_0,\ \left\| x_t(\alpha,\varphi_n) \right\| < \delta_0,\ t \in \left[\alpha, t_n^{\delta_0}\right) \quad (4)$$

It is obvious, that $t_n^{\delta_0} \to \infty$.

Due to condition $V(t,0) \equiv 0$ one can assume that there are numbers $\Delta_n \to 0$, so that $V(\alpha,\varphi_n) \le \Delta_n$. As $\dot{V}(t,\varphi) \le 0$, we get:

$$V(t, x_t^n(\alpha,\varphi_n)) \le \Delta_n, t > \alpha \quad (5)$$

We put $\delta_1 = \dfrac{\delta_0}{2}$. Suppose $T = T(\delta_1)$ is a number, defined from condition 2 of the theorem according to definition 3. Let $t_n^{\delta_1} = t_n^{\delta_0} - T$.

From Assumption 1 the family of function $\left\{ x_{t_n^{\delta_1}}^n(\alpha,\varphi_n) \right\}$ is precompact in C_{δ_0}, than there is the presequence (without restricting of the generality we accept that it coincides with $t_n^{\delta_1}$) and the function φ_{δ_1} are so, that $x_{t_n^{\delta_1}}^n \to \varphi_{\delta_1}$, where $\left\|\varphi_{\delta_1}\right\| \le \delta_{0.}$ In this case owing to (5) and to condition for the derivative V we have: $V(t_n^{\delta_1}, x_{t_n^{\delta_1}}^n(\alpha,\varphi_n)) \to 0$ where $n \to \infty$, it means that $\varphi_{\delta_1} \in V^{-1}(\infty,0)$.

It follows from (4) that:

$$\left\| x_{t_n^{\delta_1}+T}(t_n^{\delta_1}, x_{t_n^{\delta_1}}(\alpha,\varphi_n)) \right\| = \delta_0, \qquad \left\| x_{t_n^{\delta_1}}(t_n^{\delta_1}, x_{t_n^{\delta_1}}(\alpha,\varphi_n)) \right\| < \delta_0. \quad (6)$$

Therefore, there is a subsequence (without restricting of the generality that it coincides with $t_n^{\delta_1}$), such that $f(t + t_n^{\delta_1}, \varphi) \to f_{\delta_1}^*(t,\varphi)$. According to theorem 1 we obtain that $x^n(t + t_n^{\delta_1}, \alpha, \varphi_n) \to x^*(t,0,\varphi_{\delta_1})$ where $x^*(t,0,\varphi_{\delta_1})$ is the solution of equation $\dot{x}(t) = f_{\delta_1}^*(t, x_t)$.

But then by convention of number T we have that $\left| x^*(t,0,\varphi_{\delta_1}) \right| \le \delta_1$ for each $t \ge T$. Taking the limiting process we obtain from relation of (6), that $\left\| x_T^*(0,\varphi_{\delta_1}) \right\| = \delta_0 = 2\delta_1$. We deduced the contradiction with the choice of the number of T. Thus we have a stability of zero solution (1).

The theorem is proved.

Theorem 3. We will assume that in addition to the conditions of Theorem 2, $V(t,\varphi) \le \omega(\|\varphi\|)$ is realized, then the solution $x = 0$ (1) is uniformly stable.

The proof is obtained analogous to the proof of Theorem 2.

Analogous to the proof of Theorem 2 with certain alterations we may proof the following Theorem.

Theorem 4. We will assume that:

1) the continuous functional exists $V: R^+ \times C_H \to R^+$ such that
$0 \le V(t,\varphi) \le \omega(\|\varphi\|), V(t,0) \equiv 0, \dot{V}(t,\varphi) \le 0,$
$(t,\varphi) \in R^+ \times C_H$
2) the solution x=0 is asymptotically stable uniformly with respect to the set $\Lambda_0 = V^{-1}(\infty,0)$.

Then the solution x=0 of equation (1) is uniformly stable by Lyapunov.

3. Example

Consider the following system of equations:

$$\begin{cases} \ddot{x}_1 = -p(t, x_1(t - r_1(t)), \dot{x}_1(t - r_2(t)))\dot{x}_1(t) - f_1(t, x_1(t - r_3(t))) + \\ + g(t, x_1(t), \dot{x}_1(t), x_2(t), \dot{x}_2(t)), \\ \ddot{x}_2 = -f_2(x_2(t)). \end{cases} \tag{7}$$

We will admit that $f_1(t,0) \equiv 0,\ f_2(0) = 0, r_i(t): 0 < r_i(t) \le h = const$

We will assume that the function $f_1(t,x)$ has a continuous partial derivative to x and functions $p(t, x_1, \dot{x}_1),\ f_1(t, x_1),\ f_{1x_1}(t, x_1), r_i(t)$ are confined, uniformly continuous functions of their arguments $t \in R^+, x_1, x_2 \in [-\beta, \beta] (\beta > 0)$, functions $p(t, x_1, \dot{x}_1),\ f_1(t, x_1),\ f_{1x_1}(t, x_1)$ satisfies for $x_1, x_2 \in [-\beta, \beta]$ Lipschitz conditions:

$$\left| f(t, x_{12}) - f(t, x_{11}) \right| \le l \left| x_{12} - x_{11} \right| (l = const)$$

We will assume that the following conditions are met:

1) for any $\varepsilon > 0$ we may find $\delta = \delta(\varepsilon) > 0$ such that for $\{x_1 : |x_1| = \varepsilon\}$ the inequality is satisfied:

$$\lim_{t \to +\infty} \inf \left| f(t, x_1) \right| \ge \delta \tag{8}$$

2) for all $t \in R^+, x_1 \in [-\beta, \beta]$:

$$t \in R^+, x_1 \in [-\beta, \beta] \tag{9}$$

3) $$f_1(t, x_1) x_1 > 0 (\forall t \ge 0, \forall x_1 : |x_1| > 0) \tag{10}$$

$$f_2(x_2) x_2 > 0 (\forall x_2 : |x_2| > 0) \tag{11}$$

4) for $t \in R^+, x_1, x_2 \in [-\beta, \beta]$:

$$\left| f_{1x_1}(t,x_1) \right| \le L = const \tag{12}$$

$$p(t,x_1,\dot{x}_1) \ge h\mu_0 = const > 0, \mu_0 > L \tag{13}$$

5) $$g(t,x_1,\dot{x}_1,0,0) \equiv 0 \tag{14}$$

Proof that undisturbed motion $\dot{x}_1 = \dot{x}_2 = x_1 = x_2 = 0$ of system (7) is uniformly stable.

From the motion equation to x_2 find the first integral:

$$U_0(x_2,\dot{x}_2) = \frac{\dot{x}_2^2(t)}{2} + \int_0^{x_2(t)} f_2(\tau)d\tau = c = const$$

Let us take $U_0(x_2,\dot{x}_2)$ in place of Lyapunov functional *V*. It is obvious that this functional allows for an infinitely small upper limit, it is constant signs and its derivate is $\dot{V} \equiv 0$.

On the set:

$$V^{-1}(\infty,0) = U_0^{-1}(\infty,0) = \{U_0(x_2,\dot{x}_2) = 0\} = \{x_2 = 0, \dot{x}_2 = 0\}$$

the first equation (7) takes on form:

$$\ddot{x}_1 = -p(t,x_1(t-r_1(t)),\dot{x}_1(t-r_2(t)))\dot{x}_1(t) - f_1(t,x_1(t-r_3(t))) \tag{15}$$

Research the solution stability of $x_1 = \dot{x}_1 = 0$ of equation (15). For convenience we will mark:

$$x_1 = y_1 \qquad \dot{x}_1 = y_2$$

Then the equation (15) may be written in system form:

$$\begin{cases} \dot{y}_1(t) = y_2(t), \\ \dot{y}_2(t) = -p(t,y_1(t-r_1(t)),y_2(t-r_2(t)))y_2(t) - \\ - f_1(t,y_1(t)) + \int_{-r_3(t)}^{0} f_{1y_1}(t,y_1(t+s))y_2(t+s)ds. \end{cases} \tag{16}$$

For researching stability of zero solution (16) we will consider the following Lyapunov functional:

$$V_1(t,\varphi_1,\varphi_2) = \frac{\varphi_2^2(0)}{2} + \int_0^{\varphi_1(0)} f_1(t,s)ds + \frac{\mu_0}{2}\int_{-h}^{0}\int_{s}^{0} \varphi_2^2(u)duds.$$

Here, $\varphi_1(0) = y_1(t), \varphi_2(s) = y_2(t+s), -h \le s \le 0$

The functional V_1 allows for an infinitely small upper limit and it is constant sign to $\varphi(0) = (\varphi_1,\varphi_2)$ that follows from the functional and the above conditions.

For derivative $\dot{V}_1(t,\varphi)$ in consequence of (16), if the conditions (8) –

(13) have been fulfilled we find an estimate:

$$\dot{V}_1(t,\varphi) \le -\frac{\mu_0 - L}{2}\int_{-h}^{0}(\varphi_2^2(0)+\varphi_2^2(s))ds \le 0.$$

The limit to (16) system is the following:

$$\begin{cases} \dot{y}_1(t) = y_2(t), \\ \dot{y}_2(t) = -p^*(t, y_1(t - r_1^*(t)), y_2(t - r_2^*(t)))y_2(t) - \\ - f_1^*(t, y_1(t)) + \int_{-r_3^*(t)}^{0} f_{1y_1}^*(t, y_1(t+s))y_2(t+s)ds \end{cases} \tag{17}$$

We put $W_1 = \frac{\mu_0 - L}{2}\int_{-h}^{0}(\varphi_2^2(0)+\varphi_2^2(s))ds$. We get the set $\{W_1^*(t,\varphi_2)=0\} \equiv \{\varphi_2 = 0\}$. Substituting the value $y_2(t) \equiv 0$ in the system (17), we find out that the set $\{W_1^*(t,\varphi)=0\}$ doesn't contain the whole solution of the system (17), except zero. According the theorem 4.5 from (4) we get uniform asymptotic stability $y_2 = y_1 = 0$ of system (16), it means also we get uniform asymptotic stability of zero solution $\dot{x}_1 = x_1 = 0$ of system (15). Then, the solution $\dot{x}_1 = \dot{x}_2 = x_1 = x_2 = 0$ of system (7) is asymptotically stable to the set:

$$V^{-1}(\infty,0) = U_0^{-1}(\infty,0) = \{U_0(x_2,\dot{x}_2)=0\} = \{x_2 = 0, \dot{x}_2 = 0\}.$$

By this means, according to the theorem 4, we get the uniform stability of unperturbed motion $\dot{x}_1 = \dot{x}_2 = x_1 = x_2 = 0$ of system (7).

4. Conclusion

There is the development of the method of Lyapunov constant -sign functionals with using of the limit equations in the work. The obtained theorems 2-4 develop and expand some results from [4].

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Received: August 5, 2014