On the Method of Lyapunov Constant-Sign Functionals

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Abstract

In the work there is investigated the stability of the zero solution of a non-autonomous functional differential equation of the delayed type by means of limiting equations and Lyapunov constant-sign functional. An appropriate illustrating example is given.

Keywords: Lyapunov constant-sign functional, limit equations, stability problem

1. Introduction. Basic definitions and limiting equations

Suppose $\mathbb{R}$ is a real axis, $\mathbb{R}^n$ is a real linear space of n-vectors $x$ with a norm $|x|$, $h = \text{const} > 0$ is a real number, $C$ is the Banach space of continuous
functions $\varphi : [-h,0] \to \mathbb{R}^n$ with a norm $\|\varphi\| = \sup(|\varphi(s)|, -h \leq s \leq 0)$. $C_H$ is a space $\{\varphi \in C : \|\varphi\| < H, H > 0\}$. For a continuous function $x : ]-\infty, +\infty[ \to \mathbb{R}^n$ and every $t \in \mathbb{R}$, the function $x_t(s) \in C_H$ is defined by the equality $x_t(s) = x(t + s), s \in [-h,0]$. A right-hand derivative is denoted by $\dot{x}(t)$.

The functional differential equation with a finite delay

$$\dot{x}(t) = f(t, x_t), f(t,0) = 0$$

(1)

is considered, where $f : \mathbb{R}^+ \times C_H \to \mathbb{R}^n$ is a continuous function which satisfies the following assumptions [1, 4].

Assumption 1. The function $f = f(t, \varphi)$ is bounded in each set $\mathbb{R}^+ \times \overline{C}_r, \overline{C}_r = \{\varphi \in C : \|\varphi\| = r < H\}$:

$$|f(t, \varphi)| \leq m(r), (t, \varphi) \in \mathbb{R}^+ \times \overline{C}_r.$$

The smoothing of the solutions of Eq. (1) as $t$ increases follows from this assumption and, in particular, if $x = x(t, \alpha, \varphi), (\alpha, \varphi) \in \mathbb{R}^+ \times C_H$ is a solution of Eq.(1) which satisfies the initial condition $x_{\alpha} = \varphi$, then, for the values $t \geq \alpha + h$, the function $x_t \in \Gamma$, where $\Gamma \subset C_H$ is the union of the family of imbedded compact sets $\Gamma = \bigcup_{n=1}^{\infty} K_n, K_1 \subset K_2 \subset \ldots \subset K_n \subset \ldots [1]$.

Assumption 2. The function $f = f(t, \varphi)$ satisfies the Lipschitz condition with respect to $\varphi$ in each compact set $K \subset C_H$, that is for any $t \in \mathbb{R}^+$ and $\varphi_1, \varphi_2 \in K$ the inequality holds:

$$|f(t, \varphi_1) - f(t, \varphi_2)| \leq l(K)\|\varphi_1 - \varphi_2\|.$$ 

The uniqueness of the solution $x = x(t, \alpha, \varphi), (\alpha, \varphi) \in \mathbb{R}^+ \times C_H$ of Eq.(1) follows from the second assumption [3].

Assumption 3. The function $f = f(t, \varphi)$ is uniformly continuous with respect to $(t, \varphi) \in \mathbb{R}^+ \times K$ for each compact set $K \subset C_H$.

Under this assumption, 3 the family of shifts $\{f^\tau(t, \varphi) = f(t + \tau, \varphi), \tau \in \mathbb{R}^+\}$ is precompact in a certain space $F$ of continuous functions $f : \mathbb{R}^+ \times \Gamma \to \mathbb{R}^n$ [1].
A function \( f^* : \mathbb{R}^+ \times \Gamma \to \mathbb{R}^n \) is called as the limiting function to \( f(t, \varphi) \), if a sequence \( t_n \to +\infty \) exists such that \( f(t + t_n, \varphi) \) is convergent to \( f^*(t, \varphi) \) in \( F \).

The equation

\[
\dot{x}(t) = f^*(t, x)
\]

(2)
is called the limiting one to (1). The domain of the limiting equation is defined by \( \mathbb{R} \times \Gamma \). The uniqueness of solution of Eq. (2) follows from the second assumption.

The correlation between the solutions of Eq.(1) and Eq.(2) is determined by the following theorem [1].

**Theorem 1.** Suppose the function \( f : \mathbb{R}^+ \to \mathbb{R}^n \) is a limiting function to \( f \) in \( F \) with respect to a sequence \( t_n \to +\infty \), and sequences \( \{\alpha_n \in \mathbb{R}^+\} \) and \( \{\varphi_n \in \Gamma\} \) are such, that \( \alpha_n \to \alpha \in \mathbb{R}^+ \), \( \varphi_n \to \varphi \in \Gamma \) at \( n \to \infty \). Then, if \( x = x(t_n + \alpha_n, \varphi_n) \) are solutions of Eq.(1), and \( x^* = x^*(t, \alpha, \varphi) \) is a solution of limiting Eq.(2), defined for \( t \in [\alpha - h, \beta] \), the sequence of functions \( x = x(t + t_n, t_n + \alpha_n, \varphi_n) \) converges to \( x^* = x^*(t, \alpha, \varphi) \) uniformly with respect to \( t \in [\alpha - h, \gamma] \) for every \( \gamma < \beta \).

**2. Basic results. Stability theorems**

We will investigate the problem of the stability on the base of Lyapunov constant-sign functionals. We shall use the following definitions.

**Definition 1.** The solution \( x = 0 \) of Eq.(1) is stable with respect to set \( \Lambda \subset C_H \), if, for any \( \varepsilon > 0 \) one can get \( \delta = \delta(\varepsilon) > 0 \), so that for \( \varphi \in \Lambda \cap \{\|\varphi\| < \delta\} \) it is true that \( |x(t, 0, \varphi)| < \varepsilon \) for each solution \( x(t, 0, \varphi) \) of Eq.(1) for any \( t \geq 0 \).

**Definition 2.** The solution \( x = 0 \) of Eq.(1) is uniformly asymptotically stable with respect to set \( \Lambda \subset C_H \), if it is stable with respect to \( \Lambda \subset C_H \) and a \( \Delta > 0 \) exists, so that for any \( \varepsilon > 0 \) one can get \( T = T(\varepsilon) > 0 \), so that for every \( \varphi \in \Lambda \cap \{\|\varphi\| < \Delta\} \) it is true that \( \|x(t, 0, \varphi)\| < \varepsilon \) for any \( t \geq T \).

**Definition 3.** The solution \( x = 0 \) is a point of uniform attraction for the whole family of limiting equations \( \{\dot{x}(t) = f^*(t, x)\} \) with respect to set \( \Lambda \subset C_H \), if a \( \Delta \) exists, so that for any \( \varepsilon > 0 \) there is \( T = T(\varepsilon) > 0 \), so that
for any solution \( x^*(t,0,\varphi) \), \( \varphi \in \Lambda \cap \{ \| \varphi \| < \Delta \} \) of any equation \( \dot{x}(t) = f^*(t,x(t)) \) for any \( t \geq T \) the inequality \( \| x^*_t(0,\varphi) \| < \varepsilon \) holds.

Suppose \( V : R^+ \times C_H \rightarrow R^+ \) is a certain continuous functional, \( x = x(t,\alpha,\varphi), (\alpha,\varphi) \in R^+ \times C_H \) is a certain solution of Eq.(1). Along this solution the functional \( V \) is a continuous time-dependent function \( V(t) = V(t,x(t,\alpha,\varphi)) \). For this function it is possible to define an upper right-hand derivative \( \dot{V}(t,\varphi) \).

Let us denote \( \omega^+_i(u) \) continuous strictly monotonically increasing functions \( \omega^+_i : R^+ \rightarrow R^+ \), \( \omega^+_i(0) = 0 \).

Definition 4. Let us define a set for the functional \( V(t,\varphi) : V^{-1}(\infty,0) = \{ \varphi \in C_H : \exists \varphi_n \in C_H, \exists \tau_n \rightarrow +\infty : \varphi_n \rightarrow \varphi, V(t_n,\varphi_n) \rightarrow 0, n \rightarrow +\infty \} \).

The definitions which have been introduced enable us to derive the sufficient conditions of stability and asymptotic stability when a non-negative functional with a non-positive derivative exists.

Let us prove the following theorem about stability of zero solution \( x = 0 \) of Eq.(1) using a constant-sign functional \( V \).

Theorem 2. Suppose that:

1) a continuous functional \( V : R^+ \times C_H \rightarrow R^+ \) exists, so that \( V(t,\varphi) \geq 0, V(t,0) = 0, \dot{V}(t,\varphi) \leq 0, (t,\varphi) \in R^+ \times C_H \);

2) the solution \( x = 0 \) is a point of uniform attraction for solutions \( \{ \dot{x}(t) = f^*(t,x(t)) \} \) with respect to the set \( \Lambda_0 = V^{-1}(\infty,0) \).

Then the solution \( x = 0 \) is stable by Lyapunov.

Proof.

We will assume that the position of equilibrium \( x = 0 \) of Eq. (1) is unstable. Then, at a certain \( \varepsilon_0 : 0 < \varepsilon_0 < H \) there is a moment \( \alpha > 0 \), a sequence \( \{ \varphi_n : \| \varphi_n \| \rightarrow 0, n \rightarrow +\infty \} \), so that for the solutions of Eq.(1) \( x^n(t) = x(t,\alpha,\varphi_n) \) the following is true:

\[
\| x^n_*(\alpha,\varphi_n) \| = \varepsilon_0
\]

at a certain \( t = t_n \). From the uniqueness of \( x = 0, t_n \rightarrow +\infty \) follows.

Suppose a \( \Delta > 0 \) is a number, defined by condition 2 of the theorem. We assume \( \delta_0 = \frac{1}{2} \min(\varepsilon_0,\Delta) \). Then for the solutions \( x^n(t) = x(t,\alpha,\varphi_n) \) there is a
sequence $t_n^\delta \leq t_n^\varepsilon$, so that:

$$
\left\| x_{t_n^\varepsilon}^{n} (\alpha, \varphi_n) \right\| = \delta_0, \left\| x_{t_n^\varepsilon} (\alpha, \varphi_n) \right\| < \delta_0, \ t \in [\alpha, t_n^\varepsilon) .
$$

(4)

It is obvious, that $t_n^\delta \to \infty$.

Due to condition $V(t,0) \equiv 0$ one can assume that there are numbers $\Delta_n \to 0$, so that $V(\alpha, \varphi_n) \leq \Delta_n$. As $\dot{V}(t, \varphi) \leq 0$, we get:

$$
V(t, x^n_t (\alpha, \varphi_n)) \leq \Delta_n, t > \alpha .
$$

(5)

We put $\delta_1 = \frac{\delta_0}{2}$. Suppose $T = T(\delta_1)$ is a number, defined from condition 2 of the theorem according to definition 3. Let $t_n^\delta = t_n^\varepsilon - T$.

From Assumption 1 the family of function $\left\{ x^n_{t_n^\delta} (\alpha, \varphi_n) \right\}$ is precompact in $C_{\delta_1}$, than there is the presequence (without restricting of the generality we accept that it coincides with $t_n^\delta$) and the function $\varphi_{\delta_1}$ are so, that $x^n_{t_n^\delta} \to \varphi_{\delta_1}$, where $\| \varphi_{\delta_1} \| \leq \delta_0$. In this case owing to (5) and to condition for the derivative $V$ we have:

$$
V(t_n^\delta, x^n_{t_n^\delta} (\alpha, \varphi_n)) \to 0 \text{ where } n \to \infty , \text{ it means that } \varphi_{\delta_1} \in V^{-1}(\infty,0) .
$$

It follows from (4) that:

$$
\left\| x_{t_n^\delta}^{n} \right\| = \delta_0, \left\| x_{t_n^\delta} (\alpha, \varphi_n) \right\| < \delta_0 .
$$

(6)

Therefore, there is a subsequence (without restricting of the generality that it coincides with $t_n^\delta$), such that $f(t + t_n^\delta, \varphi) \to f^*_\delta (t, \varphi)$. According to theorem 1 we obtain that $x^*_t (t + t_n^\delta, \alpha, \varphi_n) \to x^*_t (t,0, \varphi_{\delta_1})$ where $x^*_t (t,0, \varphi_{\delta_1})$ is the solution of equation $\dot{x}(t) = f^*_\delta (t, x_t)$.

But then by convention of number $T$ we have that $x^*_t (t,0, \varphi_{\delta_1}) \leq \delta_1$ for each $t \geq T$. Taking the limiting process we obtain from relation of (6), that $\left\| x^*_t (0, \varphi_{\delta_1}) \right\| = \delta_0 = 2 \delta_1$. We deduced the contradiction with the choice of the number of $T$. Thus we have a stability of zero solution (1).

The theorem is proved.

**Theorem 3.** We will assume that in addition to the conditions of Theorem 2, $V(t, \varphi) \leq \omega (\| \varphi \|)$ is realized, then the solution $x = 0$ (1) is uniformly stable.

The proof is obtained analogous to the proof of Theorem 2.

Analogous to the proof of Theorem 2 with certain alterations we may proof the following Theorem.
Theorem 4. We will assume that:
1) the continuous functional exists \( V : \mathbb{R}^+ \times C_H \rightarrow \mathbb{R} \) such that
\[
0 \leq V(t, \varphi) \leq \omega(\|\varphi\|) V(t, 0) = 0, \dot{V}(t, \varphi) \leq 0,
\]
\((t, \varphi) \in \mathbb{R}^+ \times C_H\)
2) the solution \( x=0 \) is asymptotically stable uniformly with respect to the set \( \Lambda_0 = V^{-1}(\infty, 0) \).

Then the solution \( x=0 \) of equation (1) is uniformly stable by Lyapunov.

3. Example

Consider the following system of equations:
\[
\begin{align*}
\dot{x}_1 &= -p(t, x_1(t) - r_1(t)), \dot{x}_2(t) = f_1(t, x_1(t) - r_1(t)) + g(t, x_1(t), x_2(t), x_2(t)), \\
\dot{x}_2 &= -f_2(x_2(t)).
\end{align*}
\]
(7)

We will admit that \( f_1(t, 0) = 0, f_2(0) = 0, r_i(t) : 0 < r_i(t) \leq h = \text{const} \)

We will assume that the function \( f_1(t, x) \) has a continuous partial derivative to \( x \) and functions \( p(t, x_1, x_2), f_1(t, x_1), f_{1i}(t, x_1), r_i(t) \) are confined, uniformly continuous functions of their arguments \( t \in \mathbb{R}^+, x_1, x_2 \in [-\beta, \beta] (\beta > 0) \), functions
\( p(t, x_1, x_2), f_1(t, x_1), f_{1i}(t, x_1) \) satisfies for \( x_1, x_2 \in [-\beta, \beta] \) Lipschitz conditions:
\[
|f(t, x_{12}) - f(t, x_{11})| \leq l|x_{12} - x_{11}| (l = \text{const})
\]

We will assume that the following conditions are met:
1) for any \( \varepsilon > 0 \) we may find \( \delta = \delta(\varepsilon) > 0 \) such that for \( \{x_1 : |x_1| = \varepsilon\} \) the inequality is satisfied:
\[
\lim_{t \to +\infty} \inf |f(t, x_1)| \geq \delta
\]
(8)
2) for all \( t \in \mathbb{R}^+, x_1 \in [-\beta, \beta] \):
\[
t \in \mathbb{R}^+, x_1 \in [-\beta, \beta]
\]
(9)
3) \( f_1(t, x_1)x_1 > 0 (\forall t \geq 0, \forall x_1 : |x_1| > 0) \)
(10)
4) \( f_2(x_2)x_2 > 0 (\forall x_2 : |x_2| > 0) \)
(11)
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\[ |f_{1_2}(t, x_1)| \leq L = \text{const} \quad (12) \]
\[ p(t, x_1, \dot{x}_1) \geq h \mu_0 = \text{const} > 0, \mu_0 > L \quad (13) \]
\[ g(t, x_1, \dot{x}_1, 0, 0) \equiv 0 \quad (14) \]

5) 

Proof: that undisturbed motion \( \dot{x}_1 = \dot{x}_2 = x_1 = x_2 = 0 \) of system (7) is uniformly stable.

From the motion equation to \( x_2 \) find the first integral:

\[ U_0(x_2, \dot{x}_2) = \frac{\dot{x}_2^2(t)}{2} + \int_{0}^{\infty} f_2(t) dt = c = \text{const} \]

Let us take \( U_0(x_2, \dot{x}_2) \) in place of Lyapunov functional \( V \). It is obvious that this functional allows for an infinitely small upper limit, it is constant signs and its derivate is \( \dot{V} \equiv 0 \).

On the set:

\[ V^{-1}(\infty, 0) = U_0^{-1}(\infty, 0) = \{ U_0(x_2, \dot{x}_2) = 0 \} = \{ x_2 = 0, \dot{x}_2 = 0 \} \]

the first equation (7) takes on form:

\[ \ddot{x}_1 = -p(t, x_1(t - r_1(t))), \dot{x}_1(t - r_2(t))) \dot{x}_1(t) - f_1(t, x_1(t - r_2(t))) \]

Research the solution stability of \( x_1 = \dot{x}_1 = 0 \) of equation (15). For convenience we will mark:

\[ x_i = y_1 \quad \dot{x}_1 = y_2 \]

Then the equation (15) may be written in system form:

\[ \begin{cases}
\dot{y}_1(t) = y_2(t), \\
\dot{y}_2(t) = -p(t, y_1(t - r_1(t))), y_2(t - r_2(t)))y_2(t) - \end{cases} \]

\[ - f_1(t, y_1(t)) + \int_{-r_1(t)}^{0} f_{1y}(t, y_1(t + s))y_2(t + s)ds. \quad (16) \]

For researching stability of zero solution (16) we will consider the following Lyapunov functional:

\[ V_1(t, \varphi_1, \varphi_2) = \frac{\varphi_2^2(0)}{2} + \int_{0}^{\varphi_1(0)} f_1(t, s)ds + \frac{\mu_0}{2} \int_{-h}^{0} \varphi_2^2(u) du ds. \]

Here, \( \varphi_1(0) = y_1(t), \varphi_2(s) = y_2(t + s), -h \leq s \leq 0 \)

The functional \( V_1 \) allows for an infinitely small upper limit and it is constant sign to \( \varphi(0) = (\varphi_1 \varphi_2) \) that follows from the functional and the above conditions.

For derivative \( \dot{V}_1(t, \varphi) \) in consequence of (16), if the conditions (8) –
(13) have been fulfilled we find an estimate:

\[
\dot{V}_1(t, \varphi) \leq -\frac{\mu_0 - L}{2} \int_{-h}^{0} (\varphi_2^*(0) + \varphi_2^*(s))ds \leq 0.
\]

The limit to (16) system is the following:

\[
\begin{cases}
\dot{y}_1(t) = y_2(t), \\
\dot{y}_2(t) = -p^*(t, y_1(t - r_1^*(t)), y_2(t - r_2^*(t)))y_2(t) - \\
- \int_{-t_1^*(t)}^{0} f^*_1(t, y_1(t + s))y_2(t + s)ds & (17)
\end{cases}
\]

We put \( W_i = \frac{\mu_0 - L}{2} \int_{-h}^{0} (\varphi_2^*(0) + \varphi_2^*(s))ds \). We get the set \( \{W_1(t, \varphi) = 0\} = \varphi_2 = 0 \}. Substituting the value \( y_2(t) = 0 \) in the system (17), we find out that the set \( \{W_1(t, \varphi) = 0\} \) doesn’t contain the whole solution of the system (17), except zero. According the theorem 4.5 from (4) we get uniform asymptotic stability \( y_2 = y_1 = 0 \) of system (16), it means also we get uniform asymptotic stability of zero solution \( \dot{x}_1 = x_1 = 0 \) of system (15). Then, the solution \( \dot{x}_1 = \dot{x}_2 = x_1 = x_2 = 0 \) of system (7) is asymptotically stable to the set:

\[
V^{-1}(\infty, 0) = U^{-1}_{0}(\infty, 0) = \{U_0(x_2, \dot{x}_2) = 0\} = \{x_2 = 0, \dot{x}_2 = 0\}.
\]

By this means, according to the theorem 4, we get the uniform stability of unperturbed motion \( \dot{x}_1 = \dot{x}_2 = x_1 = x_2 = 0 \) of system (7).

4. Conclusion

There is the development of the method of Lyapunov constant -sign functionals with using of the limit equations in the work. The obtained theorems 2-4 develop and expand some results from [4].

References


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