On the Computation of the Adjoint Ideal of Curves with Ordinary Singularities

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Abstract

Let $C$ be a projective algebraic curve. Suppose that $C$ has ordinary singularities (with reduced tangent cones). In this paper we construct an efficient algorithm for computing locally, with polynomial computational cost, the largest adjoint ideal of $C$.

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1 Introduction

Let $C$ be an algebraic curve. If $C$ is a plane affine curve, the adjoint curves of $C$ (curves passing through all singularities of a given curve with high enough multiplicity) play a fundamental role in various areas of mathematics: algebraic geometry, number theory, coding theory, etc. The adjoint curves form an ideal called the adjoint ideal of $C$. Various algorithms for computing the adjoint ideal of $C$ in the planar case have been introduced [3],[6]. The notion of adjoint curve and adjoint ideal of a plane affine curve can be extended, following classical ideas of Castelnuovo and Petri, to the notion of adjoint hypersurface of a space curve $C$ in the projective space $\mathbb{P}^n$ ([1]). There are various adjoint ideals to space curves [1]. We call the largest one $\text{Adj}(C)$. $\text{Adj}(C)$ has the property that its local ideal, in the local ring $A$ of $C$ at a singular point $Q$, is
the conductor $I$ of $A$ (in its normalization $\overline{A}$). No general algorithm for the computation of the conductor of $A$ is known. In this paper we construct an algorithm for computing $I$ when $Q$ is an ordinary singular point with reduced tangent cone. The key idea is reconstituting the algorithm to the computation of the conductor of the coordinate ring of the points of the projective tangent cone $\text{Proj}(G(A))$ to $C$ at $Q$. In particular, if the points of $\text{Proj}(G(A))$ are in generic position, $I$ is a power of the maximal ideal of $A$.

2 Adjoint ideals to a curve and conductor

In this section, for all undefined notions we refer to [5]. Let $C \subset P^n$ be an irreducible, non-degenerate complete curve of degree $d$ and genus $g$, defined over an algebraically closed field $k$. Let $\pi: \overline{C} \to C$ be the normalization morphism. We denote by $D$ the divisor on $C$, pull-back via $\pi$ of a hyperplane in $P^n$. Let $S = \bigoplus_{t \geq 0} S_t$ be the ring $k[X_0, \ldots, X_r]$ and let $G$ be any graded, saturated ideal in $S$, properly containing the ideal of polynomials vanishing on $C$. Let $\text{Sing}(C)$ be the finite set of singular points of $C$ and let $Q \in \text{Sing}(C)$. Let $I$ be the local ideal of $G$ in the local ring $A$ of $C$ at $Q$. Suppose that $I$ is a proper ideal for any $Q$ (this means that the hypersurfaces of the projective variety $V(G)$ corresponding to $G$ contain all the singular points of $C$). If $M_1, \ldots, M_h$ are the maximal ideals of the normalization $\overline{A}$ of $A$ and $I\overline{A}$ is the extension of $I$ to $\overline{A}$, one has $I\overline{A} = M_1^{a_1} \cdots M_h^{a_h}$, with $a_1, \ldots, a_h$ positive integers. Let $Q_1, \ldots, Q_h$ be the points on $C$ corresponding to $M_1, \ldots, M_h$; we can then consider the effective divisor on $C$, defined by $G$

$$\Delta(G) = \sum_{Q \in \text{Sing}(C)} (a_1 Q_1 + \cdots + a_h Q).$$

**Definition 2.1.** The ideal $G$ is an adjoint ideal of $C$ if, for $t$ large enough, the maps $\rho_t: G_t \to H^0(C, \mathcal{O}_C(tD - \Delta(G)))$ are surjective.

It is well known that there exist adjoint ideals on $C$: for example the Castelnuovo adjoint ideal and the Petri adjoint ideal [1]. In the planar case these adjoint ideals coincide [2, (3.1)]. The following crucial theorem states that all adjoint ideals are locally extended ideals.

**Theorem 2.2.** $G$ is an adjoint ideal of $C$ if and only if, for any $Q \in \text{Sing}(C)$, $I = I\overline{A}$ that is $I$ is an extended ideal of of $A$ in $\overline{A}$

**Proof.** See [2, Thm. 2.5].

**Definition 2.3.** If $S$ is a ring, the conductor of $S$ in its integral closure (normalization) $\overline{S}$ is $\text{Ann}_S(\overline{S}/S) = \{ b \in S | b\overline{S} \subset S \}$, which is clearly the largest extended ideal of $S$ in $\overline{S}$. 

Definition 2.4. \( \text{Adj}(C) \) is the largest homogeneous ideal of \( S \) such that its local ideal \( I \), at any singular point \( Q \) of \( C \), is the conductor of the local ring \( A \) of \( C \) at \( Q \) in its normalization \( \overline{A} \).

Theorem 2.5. Any adjoint ideal of \( C \) is contained in the ideal \( \text{Adj}(C) \).

Proof. See [2, Prop. 2.1 and Lemma 2.3].

Then to compute the largest adjoint ideal of a curve \( C \) it is enough to compute the conductor of the local rings of \( C \) at its singular points. This is what we will do in the next section for ordinary singularities with reduced tangent cones.

3 The conductor of the local ring \( A \) of a curve \( C \) at an ordinary singularity with reduced tangent cone

In the following \( A \) is the local ring, at a singular point \( Q \), of an irreducible projective curve \( C \) over an algebraically closed field \( k \). We denote with \( M \) the maximal ideal of \( A \) and with \( M_i \), \( i = 1, \ldots, s \) the maximal ideals of the normalization \( \overline{A} \) of \( A \). We set \( J = M_1 \cap \cdots \cap M_s \), the Jacobson radical of \( A \). If \( n \in \mathbb{N} \), \( H(A, n) = \dim_k(M^d/M^{d+1}) \) denotes the Hilbert function of \( A \) and \( e(A) \) the multiplicity of \( A \) at \( M \), that is the multiplicity of \( C \) at the singular point \( Q \) (\( H(A, d) = e(A) \), for \( n \) large enough). \( H(A, 1) \) is the embedding dimension \( \text{emdim}(A) \) of \( A \) (i.e. the dimension of the tangent space of \( C \) at \( Q \)). For any semilocal ring \( B \), the form ring of \( B \), with respect to its Jacobson radical \( N \), is \( G(B) = \bigoplus_{n \in \mathbb{N}}(N^n/N^{n+1}). \) \( \text{Spec}(G(A)) \) is the tangent cone of \( C \) at \( Q \) and is contained in the affine space \( \mathbb{A}^{r+1} \), \( r + 1 = \text{emdim}(A) \). If \( x \in B \), \( x \in N^n - N^{n+1}, n \in \mathbb{N} \), we say that \( x \) has degree \( n \) and \( x^* \in N^n/N^{n+1} \) is the image of \( x \) in \( G(B) \), i.e. the initial form of \( x \). If \( I \) is an ideal of \( B \), \( G(I) \) is the homogeneous ideal of \( G(B) \) generated by all the initial forms of the elements of \( I \). If \( G(I)_n \) is the set of homogeneous elements of degree \( n \) in \( G(I) \), then \( G(I) = (I \cap M^n)/(I \cap M^{n+1}) \). The natural homomorphisms \( M^n/M^{n+1} \rightarrow J^n/J^{n+1} \) induce a homomorphism of graded rings \( f : G(A) \rightarrow G(\overline{A}) \).

Proposition 3.1. The following conditions are equivalent:

i. \( G(A) \) is reduced
ii. \( J^n \cap A = M^n \) for any integer \( n \)
iii. \( f : G(A) \rightarrow G(\overline{A}) \) is injective]

Further if \( G(A) \) is reduced \( M \overline{A} = J \), \( e(A) = s \) and \( G(\overline{A}) \) is the normalization of \( \text{Im} f \).

Proof. See [9, Prop.1].
From now on we assume $G(A)$ reduced and $G(A) \subset G(\overline{A})$. In this case $Q$ is an ordinary singularity, the tangent cone $\text{Spec}(G(A))$ consists of $s$ simple lines $(s = e(A))$ and the projectivized tangent cone $\text{Proj}(G(A))$ consists of $s$-points $\{P_1, \ldots, P_s\}$ of the projective space $\mathbb{P}^r$. Let

$$I = \cap_{i=1}^s M_i$$

be the maximal homogeneous ideal of the points of the projective tangent cone (or, which is the same, of the lines of the affine tangent cone). Then we consider the homogeneous ring of the points $\mathbb{P}^r$ containing $P_i$ if and only if $G(I)$ is the conductor of $G(A) = G(\overline{A})$ [7, §4].

If $I$ is the conductor of $A$ in $\overline{A}$ and $G$ is the conductor of $G(A)$ in $G(\overline{A})$ we have $I = \cap_{i=1}^s M_i^{n_i}$ and $G = \cap_{i=1}^s G(M_i)^{m_i}$ [7, §4]. Taking into account that $G(I) = \cap_{i=1}^s G(M_i)^{n_i}$ the next theorem shows that $m_i = n_i$, for any $i$.

**Theorem 3.2.** The ideal $I$ is the conductor of $A$ in its integral closure $\overline{A}$ if and only if $G(I)$ is the conductor of $G(A)$ in $G(\overline{A})$.

*Proof.* See [9, Thm. 2].

With the previous result, one can refer the computation of the conductor of the local ring of a curve, with reduced tangent cone to the computation of the homogeneous ring of the points of the projective tangent cone (or, which is the same, of the lines of the affine tangent cone). Then we consider the algebraic variety consisting of a finite set of points $V = \{P_1, \ldots, P_s\} \subset \mathbb{P}^r$. Let $I = I(V) = \oplus_{d \in \mathbb{N}} I(V)_d$ be the homogeneous ideal of $V$ in the graded ring $R = k[X_0, \ldots, X_r]$. Let $S = R/I(V)$ be the homogeneous coordinate ring of $V$, $N$ be the maximal homogeneous ideal of $S$ and let $N_i$ be the maximal homogeneous ideals of the integral closure $\overline{S}$ of $S$. The conductor $I$ of $S$ can be written as $I = \cap N_i^{n_i}$ for suitable integers $n_i$ [7, §4]. From now on, we say that the Hilbert function $H(S, n) = \dim_k(N^n/N^{n+1})$ and the conductor of $S$ are the Hilbert function and the conductor of $V$.

**Theorem 3.3.** The ideal $I = \cap N_i^{n_i}$ is the conductor of $S$ in $\overline{S}$ if and only if $n_i$ is the least degree of a hypersurface of $\mathbb{P}^r_k$ containing $\{P_1, \ldots, P_s\}$ and not containing $P_i$.

*Proof.* See [7, Thm. 4.3].

The integer $n_i$ is called the degree of the conductor (or simply the degree) of $P_i$ in $V$ and denoted by $\deg_V(P_i)$. There is a way of characterizing $\deg_V(P_i)$ via Hilbert functions. Let $S_i$ be the homogeneous coordinate ring of the points $\{P_1, \ldots, P_s\} \setminus \{P_i\}$. Then:

**Lemma 3.4.** $\deg_V(P_i) = \text{Min}\{n|H(S_i, n) \neq H(S, n)\}$.

*Proof.* See [9, Lemma 5].

In the next proposition we show that $\deg_V(P_i)$ is bounded by the following integers:

$$\alpha = \text{Min}\{d|I(V)_d \neq 0\} = \text{Min}\{d|H(S, d) < (d+r)\} ; \quad \nu = \text{Min}\{d|H(S, d) = s\} .$$
Proposition 3.5.  
(i) $\alpha - 1 \leq \deg_V(P_i) \leq \nu$

(ii) if $\deg_V(P_i) = \deg_V(P_j)$, $1 \leq i, j \leq s$ then $\deg_V(P_i) = \nu$

(iii) $V$ contains at least $\nu + 1$ points $P_i$ with $\deg_V(P_i) = \nu$

Proof. See [9, Prop. 7]. \hfill \Box

Definition 3.6. A set of points $V = \{P_1, \ldots, P_s\}$ for which $\deg_V(P_i) = \nu$, for any $i$, is called a Cayley-Bacharach scheme (CB-scheme).

We refer to [4] for properties of CB-schemes. As a consequence of theorems 3.2 and 3.3 we get the conductor of the local ring $A$ of a curve:

Corollary 3.7. If $\text{Proj}(G(A)) = \{P_1, \ldots, P_s\}$ the ideal $I = \cap M_i^{n_i}$ is the conductor of $A$ in $\overline{A}$ if and only if $n_i = \deg_V(P_i)$, for any $i$. In particular if $\text{Proj}(G(A))$ is a Cayley-Bacharach scheme, then the ideal $M^\nu$ is the conductor of $A$ in $\overline{A}$.

4 Computing the Hilbert functions, the conductor and the Caylay-Bacharach property

By the results of section 3, given the local ring $A$ of a curve with reduced tangent cone $\text{Spec}(G(A))$ (that is with $G(A)$ reduced) any algorithm for computing the conductor of the points $\{P_1, \ldots, P_s\}$ of $\text{Proj}(G(A))$ gives an algorithm for computing the conductor of the local ring of the curve. In this section we construct an algorithm which computes, with polynomial computational cost, the conductor of the coordinate ring of any set of projective points. We will show that the algorithm needs only the computation of the rank of suitable matrices with entries the coordinates of the points of $\text{Proj}(G(A))$.

Let $V = \{P_1, \ldots, P_s\} \subset \mathbb{P}^r_k$ and $I(V)$ be the homogeneous ideal of $V$ in the graded ring $R = k[X_0, \ldots, X_r] = \oplus_{d \in \mathbb{N}} R_d$ ($R_d = \{f \in R \mid f \text{ has degree } d\}$). We have $I(V) = \oplus_{d \in \mathbb{N}} I(V)_d$, where $I(V)_d = \{f \in R_d \mid f(P_i) = 0, \ i = 1, \ldots, s\}$. Let $S = R/I(V)$ be the homogeneous coordinate ring of $V$ and $N$ be the maximal homogeneous ideal of $S$. The vector space $I(V)_d$ is easily described as the null space of a matrix with elements in $k$. In fact:

$$I(V)_d = \{f \in R_d | f(P_i) = 0 \ for \ i = 1, \ldots, s\}.$$

If we denote with $T_i^d$, $i = 1, \ldots, N(d) = \binom{d+r}{r}$, the terms of degree $d$ in the indeterminates $X_0, \ldots, X_r$, then $\{T_i^d\}_{i=1,\ldots,N(d)}$ is a basis of the $k$-vector space $R_d$. Let $G_d^N$ be the $N(d) \times s$ matrix $G_d^N = (t_{ij}^d)$, where $t_{ij}^d = T_i^d(P_j)$. We denote
the $i$-th row of $G_d^s$ with $t_i^d$ and with $t_i^d x_h$, $h = 0, \ldots, r$, the row of $G_{d+1}^s$ whose elements are $t_i^d x_{hj}$ where $x_{hj}$ is the $h$-th coordinate of $P_j$ (note that any row of $G_{d+1}^s$ can be written in the form $t_i^d x_h$ for suitable $x_h$). We denote with $p_1, \ldots, p_s$ the column-vectors of $G_d^s$ corresponding to the points $P_1, \ldots, P_s$. If $f = \sum_{i=1}^{N(d)} \lambda_i t_i^d \in R_d$, we set:

$$(G_d^s)^t f = (G_d^s)^t \left( \begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_{N(d)} \end{array} \right)$$

where $(G_d^s)^t$ denotes the transpose of $G_d^s$. Then $I(V)_d = \{ f \in R_d | (G_d^s)^t f = 0 \}$, i.e. $I(V)_d$ is the null space of the matrix $G_d^s$. Hence, for any $d \in \mathbb{N}$ we have

(4.1) $H(S, d) = N(d) - \text{dim}_k I(V)_d = \rho(G_d^s) \leq \text{Min}\{ s, N(d) \}$ \hspace{0.5cm} ($\rho = \text{rank}$),

and the integers $\alpha$ and $\nu$ of Proposition 3.5 are given by

(4.2) $\alpha = \text{Min}\{ d \mid \rho(G_d^s) < N(d) \}$, $\nu = \text{Min}\{ d \mid \rho(G_d^s) = s \}$.

In the sequel we consider matrices in reduced row-echelon form, i.e. the first non zero entry of a row is further to the left of the first non zero entry of all successive rows and each column that contains a first non zero entry has zeros everywhere else. We recall from elementary linear algebra that every matrix $D$ (with $m$ rows and $n$ columns or simply $m \times n$) can be transformed, by the elementary row operations of Gauss algorithm, in a matrix $\overline{D}$ ($m \times n$) in reduced row-echelon form. The number of non null rows of $\overline{D}$ is the rank of the matrix $D$.

We want to construct submatrices $H_d^s$ of the matrices $G_d^s$ which allow us to compute the conductor of $V = \{ P_1, \ldots, P_s \}$ in polynomial time.

(4.3) **Construction of the matrix $H_d^s$** (By induction)

If $d \leq \alpha$ then $H_d^s = G_d^s$.

If $d \geq \alpha$, let $H_d^s \subseteq G_d^s$ be constructed; we want to construct $H_{d+1}^s$. Compute the rank $\rho$ of $H_d^s$ and let $t_j^d, j = 1, \ldots, \rho$ be the rows of $G_d^s$ which correspond to the non null rows of $\overline{H_d^s}$ (they can be immediately recovered from $\overline{H_d^s}$ by taking care of the permutations needed to row-reduce $H_d^s$). Then $H_{d+1}^s$ is the matrix whose rows are the rows $t_j^d x_h$, $j = 0, \ldots, \rho$, $h = 0, \ldots, r$, of $G_{d+1}^s$ (i.e. obtained by deleting in $G_{d+1}^s$ the rows different from $t_j^d x_h$).

We define the matrix $H_{d+1}^s$ as the submatrix of $H_d^s$ obtained deleting column $i$.

**Lemma 4.1.** $\rho(H_{d+1}^s) = \rho(H_d^s)$.

**Proof.** Since $H_d^s \subseteq G_d^s$ then $\rho(H_d^s) \leq \rho(G_d^s)$. Suppose $\rho(H_d^s) = \rho(G_d^s)$. We want to prove that $\rho(H_{d+1}^s) = \rho(G_{d+1}^s)$. If $\rho(H_d^s) = \rho(G_d^s) = \rho$ then, by elementary linear algebra, there are $E = \{ t_1^d, \ldots, t_\rho^d \}$ linearly independent rows such that the other rows $t_i^d$ of $G_d^s$ are linearly dependent from $E$. Since a row of $G_{d+1}^s$ is of the form $t_i x_h$ for suitable $x_h$ then $t_i x_h$ is a linear combination of the rows $t_j^d x_h$ of $H_{d+1}^s$. This proves that $\rho(H_{d+1}^s) = \rho(G_{d+1}^s)$.

\hfill \Box
Now let $S_i$ be the homogeneous coordinate ring of the points $\{P_1, ..., P_s\} - \{P_i\}$. Then by (4.1) and Lemmas 3.4, 4.1 and
\begin{equation} \text{deg}_V(P_i) = \min\{d \mid \rho(H^s_d) \neq \rho(H^s_d)\} \end{equation}
the property $\rho(H^s_d) \neq \rho(H^s_d)$ can be easily decided, for any $i$, by inspection on $H^s_d$, by the following Lemma whose proof is elementary.

**Lemma 4.2.** Let $i_1, ..., i_h$ be the columns containing the first non zero entries of the rows of $H^s_d$. If $i \in \{1, ..., N(d)\} - \{i_1, ..., i_h\}$ then $\rho(H^s_d) \neq \rho(H^s_d)$. If $i = n$ and $n < h$ then $\rho(H^s_d) \neq \rho(H^s_d)$ if and only if the $n$-th row of $(H^s_d)$ is null.

With the previous results one can construct the following algorithm.

\begin{enumerate}
\item Set $d_0 = 0$,
\item Reset $d = d + 1$
\item Construct $H^s_d$ and compute $\rho(H^s_d)$ (by row-reduction). Set $\rho(H^s_d) = H(S,d)$.
\item If $\rho(H^s_d) < N(d)$ then set $d = \alpha$ (see 4.2)
\item If $\rho(H^s_d) \neq \rho(H^s_d)$ (Lemma 4.2) and $\rho(H^s_d) = \rho(H^s_d)$, for $i = i_1, ..., i_h$ then set $\text{deg}_V(P_{i_1}) = \ldots = \text{deg}_V(P_{i_h}) = d$ (see 4.4)
\item If $\rho(H^s_d) = s$ then set $d = \nu$ (see 4.2) and continue, else go to Step 2.
\item If $\text{deg}_V(P_i) = \nu$, for any $i = 1, ..., s$, then set: $V$ is a CB-scheme, otherwise set: $V$ isn’t a CB-scheme.
\end{enumerate}

Now we examine the computational cost of algorithm.

**Proposition 4.3.** All the matrices involved in algorithm (2.9) have dimension $m \times s$ where $m \leq (r+1)s$. Then the cost of computing their rank is given by a polynomial in $r$ and $s$ of order $O(rs^3)$. The total computational cost of the algorithm is given by a polynomial of order at most $O(rs^4)$.

**Proof.** The matrices $(G^s_d)$ have dimension $N(d) \times s$ ($N(d) = \binom{d+r}{r}$). Then, if $d \leq \alpha - 1$, $\rho(H^s_d) = \rho(G^s_d) = N(d) \leq s$ and $H^s_d$ has dimension at most $s \times s$. If $d = \alpha$, observe that $H^s_d$ has dimension at most $s \times s$. If $d = \alpha$, observe that $N(\alpha) \leq (r+1)N(\alpha - 1)$ and then $G^s_d$ has dimension $N(\alpha) \times s$, where $N(\alpha) \leq (r+1)s$. If $d > \alpha$, then, by construction $\rho(H^s_d)$, has dimension $\rho(H^s_d)(r+1) \times s$ and $\rho(H^s_d) \leq s$. Hence the first statement is proved. The second statement follows from the well known fact that the number of operations (sums and products) necessary to compute the rank of a matrix $m \times n$, if $m \geq n$, is given by a polynomial in $m$ and $n$ of order $O(n^2m)$. The third statement follows immediately from the fact that $\nu \leq s$. □
**Remark 4.4** The algorithm has been implemented in the software [10] and it reveals very fast if the points of \( \text{Proj}(G(A)) \) are in generic position [7]. In this case \( \text{Proj}(G(A)) \) satisfies the CB property and then the conductor of \( A \) is a power of the maximal ideal.

**References**


[10] Points (software for computation on points), freely available at [http://wpage.unina.it/cioffifr/EPoints.html](http://wpage.unina.it/cioffifr/EPoints.html),

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