Well-posed Fixed Point Problems
in Banach Spaces
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Abstract
We analyze different types of well-posedness for fixed point problems giving metric characterizations and sufficient conditions.

Mathematics Subject Classification: 47H10, 49K40

Keywords: Fixed point, Set-valued map, Well-posedness

1 Introduction
Set-valued maps have been widely studied in the last decades from many points of view [1], [16], [5] due to numerous applications in several fields: Mathematics [4], [10], [3].., Economy [19], [18], [2].., Engineering [20], [17]...
If $V$ is a Banach space, $K$ is a nonempty and closed subset of $V$ and

$$F : u \in K \rightarrow F(u) \subseteq K$$

is a set-valued map from $K$ to $K$, one of the most investigated topics is the fixed point problem:

$$\exists \ u \in K \text{ such that } u \in F(u)$$

that extends to Set-valued Analysis the classical fixed point problem in Functional Analysis:

$$\forall \ u \in K \text{ such that } f(u) = u$$
where \( f : u \in K \to f(u) \in K \) is a function from \( K \) to \( K \).

For many years the investigation of fixed points mainly consisted in proving existence theorems in different settings and under various assumptions (see, for example, the exhaustive book by Dugundji and Granas [9]). Recently, fixed point problems have been investigated also from different points of view and several researches have been devoted to study their well-posedness properties [15], [14] also motivated by possible applications to fractals construction [6].

We recall that, given a function \( f : K \subseteq V \to \mathbb{R} \), the minimum problem
\[
(P) \quad \min_{u \in K} f(u)
\]
is said to be Tikhonov well-posed whenever:

i) there exists a unique minimum point \( u_o \)

ii) every minimizing sequence \( (u_n)_n \), i.e. such that \( \lim_{n} f(u_n) = \min_{u \in K} f(u) \), converges to \( u_o \).

The extension of this definition to other variational problems (variational inequalities [12], quasi-variational inequalities [11], vector quasi-variational inequalities [13], Nash equilibrium problems [15]) requires the definition of suitable sequences which play the role of minimizing sequences in optimization and that can be reasonably called approximating [12], [11].

Tikhonov well-posedness of \((P)\) can be metrically characterized if \( f \) is lower semicontinuous and bounded from below: indeed, it can be proved (see [7]) that \((P)\) is well-posed if and only if
\[
\lim_{\varepsilon \to 0} \text{diam}(M_\varepsilon) = 0 \tag{1}
\]
where, for a given positive real number \( \varepsilon \),
\[
M_\varepsilon = \left\{ u \in V : f(u) \leq \inf_{v \in V} f(v) + \varepsilon \right\}
\]
is the set of the \( \varepsilon \)-minima of the function \( f \) and \( \text{diam}(H) = \sup_{u, v \in H} ||u - v|| \) denotes the diameter of a set \( H \subseteq V \). Therefore, in order to give similar characterizations for other well-posed problems, suitable sets of approximate solutions, which play the role of \( \varepsilon \)-minima in optimization, will be defined further on.

The rest of the paper is organized as follows: in Section 2 we recall useful properties of set-valued maps, illustrating them with examples, and we present two different well-posedness notions for fixed point problems. In Section 3 we prove metric characterizations of such well-posedness concepts, in line with (1), as well sufficient conditions and we emphasize that they hold without any convexity assumption on the set-valued map \( F \).
2 Preliminary Notes

If \((X, \tau)\) be a topological space we denote by \(\overline{H}\) and \(\text{int}(H)\), the closure and the interior of a non-empty set \(H \subseteq X\).

If \(V\) is a real Banach space and \(F : u \in K \subseteq V \to F(u) \subseteq K\) is a set-valued map, we denote respectively:
- by \(s\) and \(w\), the strong and the weak topology on \(V\),
- by \(\text{Fix} F = \{u \in K : u \in F(u)\}\), the set of the fixed points of \(F\);
- by \(S\text{Fix} F = \{u \in K : \{u\} = F(u)\}\) the set of the strict fixed-points of \(F\).

**Definition 2.1.** [1] A set-valued map \(F\) from \((X, \tau)\) to a topological space \((Y, \sigma)\) is said to be respectively:
- \(\sigma\)-closed-valued if the set \(F(x)\) is non-empty and \(\sigma\)-closed, for every \(x \in X\);
- sequentially \((\tau, \sigma)\)-lower semicontinuous \((\tau, \sigma)\)-lower semicontinuous for brevity) if for every \(x \in X\), every sequence \((x_n)_n\) \(\tau\)-converging to \(x\) and every \(y \in F(x)\), there exists \((y_n)_n\) \(\sigma\)-converging to \(y\) such that \(y_n \in F(x_n)\) for \(n\) sufficiently large;
- sequentially \((\tau, \sigma)\)-closed \(((\tau, \sigma)\)-closed for brevity) if for every \(x \in X\) and every sequence \((x_n)_n\) \(\tau\)-converging to \(x\), if \((y_n)_n\) \(\sigma\)-converges to \(y\) and \(y_n \in F(x_n)\) for every \(n \in \mathbb{N}\), then \(y \in F(x)\);
- sequentially \((\tau, \sigma)\)-subcontinuous \(((\tau, \sigma)\)-subcontinuous for brevity) if for every sequence \((x_n)_n\) \(\tau\)-converging in \(X\), every sequence \((y_n)_n\), such that \(y_n \in F(x_n)\), has a \(\sigma\)-convergent subsequence.

**Example 2.2.** Let \(V = \mathbb{R}\) and \(K = [0, +\infty[\). The set-valued map \(F\) defined on \(K\) by
\[
F(u) = [0, ue^{-u}]
\]
is lower semicontinuous, closed and subcontinuous.

The set-valued map \(G\) defined on \(K\) by
\[
G(u) = [0, 1/u] \quad \text{if } u > 0 \quad \text{and } G(0) = [0, +\infty[\]
is closed, lower semicontinuous but it is not subcontinuous.

The set-valued map \(H\) defined on \(K\) by
\[
H(u) = [0, u] \quad \text{if } u > 0 \quad \text{and } H(0) = [0, 1]\]
is closed, subcontinuous but it is not lower semicontinuous.

**Example 2.3.** Let \(K\) be the unitary ball of an infinite dimensional Hilbert space \(V\).

The set-valued map \(F\) defined on \(K\) by
\[
F(u) = \{v \in K : ||v|| \leq ||u||\}
\]
is \((s,s)\)-lower semicontinuous, \((s,w)\)-closed and \((s,w)\)-subcontinuous but it is not \((s,s)\)-subcontinuous. Indeed, if \((u_n)\) is a strongly convergent sequence of the unitary ball such that \(\|u_n\| = 1\) and \((v_n)\) is a sequence which weakly converges to 0 and such that \(\|v_n\| = 1\), one has \(v_n \in F(u_n)\) and \((v_n)\) cannot have any strongly convergent subsequence.

**Definition 2.4.** [1] The **Hausdorff distance** between two non-empty bounded subsets \(H\) and \(H'\) of \(V\) is:
\[
h(H, H') = \max \left\{ \sup_{u \in H} d(u, H'), \sup_{w \in H'} d(w, H) \right\}
\]
where \(d(u, H) = \inf_{v \in H} \|u - v\| = d(H, u)\) is the distance between the point \(u\) and the set \(H\).

**Definition 2.5.** A sequence \((u_n)\) is said to be \(d\)-approximating for the fixed point problem \((\mathcal{F})\) if:
\begin{enumerate}
\item \(u_n \in K\) for every \(n \in \mathbb{N}\),
\item there exists a sequence of positive real numbers \((t_n)\) decreasing to 0 such that \(d(u_n, F(u_n)) \leq t_n\) for every \(n \in \mathbb{N}\).
\end{enumerate}

**Definition 2.6.** A sequence \((u_n)\) is said to be \(h\)-approximating for the fixed point problem \((\mathcal{F})\) if:
\begin{enumerate}
\item \(u_n \in K\) for every \(n \in \mathbb{N}\),
\item there exists a sequence of positive real numbers \((t_n)\) decreasing to 0 such that \(h(\{u_n\}, F(u_n)) \leq t_n\) for every \(n \in \mathbb{N}\).
\end{enumerate}

We point out that every \(h\)-approximating sequence is also \(d\)-approximating, since \(d(u_n, F(u_n)) \leq h(\{u_n\}, F(u_n))\) for every \(n \in \mathbb{N}\). However, the converse may fail to be true as shown by the following example.

**Example 2.7.** Let \(V = \mathbb{R}\), \(K = [0, +\infty[\) and consider the set-valued map \(F\) defined on \(K\) by \(F(u) = [u, 1]\) if \(u \in [0, 1]\) and \(F(u) = [1, 2u - 1]\) if \(u \in ]1, +\infty[\). It is easy to check that for every \(u \in K\) \(d(u, F(u)) = 0\) whereas \(h(\{u\}, F(u)) = |u - 1|\). Then, a sequence is \(h\)-approximating if and only if it converges to 1, whereas every sequence in \(K\) is \(d\)-approximating.

**Definition 2.8.** The fixed point problem \((\mathcal{F})\) is said to be well-posed with respect to \(d\) if:
\begin{enumerate}
\item \(\text{Fix} F = \{u_o\}\),
\item every \(d\)-approximating sequence for \((\mathcal{F})\) converges to \(u_o\).
\end{enumerate}

The above definition has been first considered in [11], where well-posedness of quasi-variational inequalities has been investigated (indeed, a quasi-variational inequality with identically zero operator amounts to a fixed point problem), and then in [6], [14].
Definition 2.9. The fixed point problem \((\mathcal{F})\) is said to be well-posed with respect to \(h\) if:

i) \(S_{\text{Fix}} F = \{u_o\}\),

ii) every \(h\)-approximating sequence for \((\mathcal{F})\) converges to \(u_o\).

The above definition has been considered in [6] and [14].

Whenever \(\mathcal{F}\) is well-posed with respect to \(d\) and the unique fixed-point \(u_o\) is a strict fixed-point, then \(\mathcal{F}\) is also well-posed with respect to \(h\) since every \(h\)-approximating sequence is also \(d\)-approximating. In Example 2.7 the problem \((\mathcal{F})\) is well-posed with respect to \(h\) because there exists a unique strict fixed-point, namely \(u_o = 1\), and any \(h\)-approximating sequence converges towards it, but it is clearly not well-posed with respect to \(d\).

Finally, we define the following approximate fixed points sets in order to obtain results in line with equivalence (1):

\[
D_{\varepsilon} = \{ u \in K : d(u, F(u)) \leq \varepsilon \} \quad H_{\varepsilon} = \{ u \in K : h(\{u\}, F(u)) \leq \varepsilon \}
\]

and we observe that:
- \(H_{\varepsilon}\) and \(D_{\varepsilon}\) are nonempty whenever \(\text{Fix} F\) is nonempty,
- \(H_{\varepsilon} \subseteq D_{\varepsilon}\),
- Example 2.7 shows that, in general, \(D_{\varepsilon}\) may be not included in \(H_{\varepsilon}\).

3 Main Results

The next result is a characterization of well-posedness in line with condition (1) of Section 1.

Theorem 3.1. If the set-valued map \(F\) is \((s, s)\)-lower semicontinuous, \((s, w)\)-closed and \((s, w)\)-subcontinuous on \(K\), then the fixed-point problem \((\mathcal{F})\) is well-posed with respect to \(h\) (resp. with respect to \(d\)) if and only if

\[
\lim_{\varepsilon \to 0} \text{diam}(H_{\varepsilon}) = 0 \quad (2)
\]

(resp. \(\lim_{\varepsilon \to 0} \text{diam}(D_{\varepsilon}) = 0\)).

Proof. Assume that condition (2) holds and let \((u_n)_n\) be an \(h\)-approximating sequence, i.e. \(u_n \in K\) and there exists a sequence of positive real numbers \((t_n)_n\) decreasing to 0 such that \(h(\{u_n\}, F(u_n)) \leq t_n\) for every \(n \in \mathbb{N}\). Since \(u_n \in H_{t_n}\) and \(\lim_{n \to \infty} \text{diam}(H_{t_n}) = 0\), the sequence \((u_n)_n\) satisfies the Cauchy condition and it must converge to \(u \in K\). We prove that \(u \in F(u)\) by showing that \(d(u, F(u)) \leq \lim_{n} d(u_n, F(u_n)) \leq \lim_{n} h(u_n, F(u_n)) = 0\), since this implies that \(u \in F(u) = F(u)\). Assume that \(d(u, F(u)) > \lim_{n} d(u_n, F(u_n))\)
and let $\eta \in \mathbb{R}$ such that $\liminf\limits_n d(u_n, F(u_n)) < \eta < d(u, F(u))$. There exists a subsequence $(u_n)_k$ and a sequence $(v_k)_k$ such that $v_k \in F(u_{n_k})$ and $||u_{n_k} - v_k|| < \eta$ for every $k \in \mathbb{N}$. Since $F$ is $(s, w)$-subcontinuous and $(s, w)$-closed, a subsequence of $(v_k)_k$, still denoted by $(v_k)_k$, $w$-converges to $v \in F(u)$ and $||u - v|| \leq \liminf\limits_k ||u_{n_k} - v_k|| \leq \eta$ that gives a contradiction. Therefore, $u$ is a fixed point for $F$. In order to prove that $u$ is a strict fixed-point, consider any $z \in F(u)$. The $(s, s)$-lower semicontinuity of $F$ implies that there exists a sequence $(z_n)_n$ $s$-converging to $z$ such that $z_n \in F(u_n)$ for $n$ sufficiently large. Therefore one has $||z_n - u_n|| \leq h(\{u_n\}, F(u_n)) \leq t_n$ and $z = \lim\limits_n z_n = \lim\limits_n u_n = u$, so the problem $(\mathfrak{F})$ is $h$-well-posed.

Conversely, assume that $(\mathfrak{F})$ is $h$-well-posed and that $(2)$ is not true. There exists a positive number $\beta$ such that $\lim\inf\limits_n \text{diam}(\mathcal{H}_{t_n}) > \beta$ for every sequence of positive real numbers $(t_n)_n$ decreasing to 0. Therefore, for every $n \in \mathbb{N}$ there exist $u_n \in \mathcal{H}_{t_n}$ and $w_n \in \mathcal{H}_{t_n}$ such that $||u_n - w_n|| > \beta$. The sequences $(u_n)_n$ and $(w_n)_n$ should converge to the unique strict fixed point of $F$ since they are $h$-approximating and this gives a contradiction. The metric characterization of well-posedness with respect to $d$ is similar and we omit it.

The next proposition shows that in finite dimensional spaces fixed point problems with unique solution in compact sets are always well-posed.

**Theorem 3.2.** Assume that $V = \mathbb{R}^k$ and that $K \subseteq \mathbb{R}^k$ is compact. If the set-valued map $F$ is lower semi-continuous and closed on $K$ and $\text{Fix } F = \{u_o\}$ (resp. $\text{SFix } F = \{u_o\}$), then the fixed-point problem $(\mathfrak{F})$ is well-posed with respect to $d$ (resp. with respect to $h$).

**Proof.** Let $(u_n)_n$ be a $d$-approximating sequence for $(\mathfrak{F})$, i.e. $u_n \in K$ and there exists a sequence of positive real numbers $(t_n)_n$ decreasing to 0 such that $d(u_n, F(u_n)) \leq t_n$ for every $n \in \mathbb{N}$. Since $K$ is compact, there exists a subsequence $(u_{n_k})_k$ which converges to $u \in K$. Therefore, as in Theorem 3.1 one proves that $d(u, F(u)) \leq \lim\inf\limits_k d(u_{n_k}, F(u_{n_k})) = 0$ and $u \in F(u) = F(u)$.

The point $u$ is a fixed-point for $F$ and has to be equal to $u_o$. If the sequence $(u_n)_n$ does not converge to $u_o$ there exists a subsequence of $(u_n)_n$ such that any its subsequence does not converge to $u_o$. However, in the first part it has been shown that any convergent subsequence of $(u_n)_n$ has to converge to $u_o$ and one gets a contradiction.

If $(u_n)_n$ is an $h$-approximating sequence, then, reasoning as in Theorem 3.1, we conclude that $u_o \in \text{SFix } F$.

We point out that the above results do not involve any convexity condition on $K$ and on $F$, so they are of different nature compared with those given in [6] and [14], and that well-posedness properties for fixed point problems having more than one solution will be considered in a further paper.
References


Received: April 11, 2014