Gluing of Posets, and Lattices of Subgroups

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Abstract

We discuss a notion of gluing for arbitrary posets which extends the standard one for lattices. We show that the gluing of two sublattices along a suitable intersection often is a lattice. This is not always the case, though; we amend an imprecise statement in the literature on this point. Next we consider some instances of the problem of determining when the gluing of some given lattices gives rise to the lattice of all subgroups of a group, and for which groups does this happen. Our main result answers these problem for repeated gluing of finite lattices whose maximal chains have length 2.

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1 Gluing

Let \((S_1, \leq_1)\) and \((S_2, \leq_2)\) be two posets, and assume that the two orderings \(\leq_1\) and \(\leq_2\) agree on \(D := S_1 \cap S_2\), that is: for all \(x, y \in D\), \(x \leq_1 y\) if and only if \(x \leq_2 y\). It is possible to simultaneously extend \(\leq_1\) and \(\leq_2\) to an ordering of the set-theoretic union \(S = S_1 \cup S_2\): the transitive closure of the union of \(\leq_1\) and \(\leq_2\) (considered as sets of pairs) is the minimal such extension. This transitive closure has a simple description and exactly preserves the orderings on \(S_1\) and \(S_2\). We omit the proof of this elementary fact.

**Lemma 1.1.** Let \(\leq\) be the transitive closure of \(\leq_1 \cup \leq_2\). Then \(\leq\) is an ordering on \(S\) and for all \(x, y \in S\) we have \(x \leq y\) if and only if one of the following
occurs: \( x \leq_1 y, x \leq_2 y \), there exists \( d \in D \) such that \( x \leq_1 d \leq_2 y \), there exists \( d \in D \) such that \( x \leq_2 d \leq_1 y \). Moreover, if \( i \in \{1, 2\} \) and \( x, y \in S_i \), we have \( x \leq y \iff x \leq_i y \).

In this construction the posets \((S_1, \leq_1)\) and \((S_2, \leq_2)\) play the same role. We are interested in a different situation, in which this symmetry is broken and, in the extended ordering the elements of \( S_1 \) not in \( S_2 \) can precede elements of \( S_2 \setminus S_1 \), but not the other way round. This is ensured by the following condition:

\[
\text{for all } d \in D \left\{ \begin{array}{l}
\text{for all } x \in S_1, (d \leq_1 x \Rightarrow x \in D), \text{ and } \\
\text{for all } x \in S_2, (x \leq_2 d \Rightarrow x \in D).
\end{array} \right. \tag{\dagger\dagger}
\]

Thus, the requirement is that \( D \) is upward-closed in \((S_1, \leq_1)\) and downward-closed in \((S_2, \leq_2)\). This makes \( D \) a convex subset in both the original posets (that is, if \( x \) and \( y \) are elements of \( D \), then every element between \( x \) and \( y \) is in \( D \)). On assuming \((\dagger\dagger)\), the transitive closure of \( \leq_1 \cup \leq_2 \) can be described by letting, for all \( x, y \in S \):

\[
x \leq y \iff \begin{cases} 
& x, y \in S_1 \text{ and } x \leq_1 y, \text{ or } \\
& x, y \in S_2 \text{ and } x \leq_2 y, \text{ or } \\
& x \in S_1 \text{ and } y \in S_2 \text{ and } x \leq_1 d \leq_2 y \text{ for some } d \in D.
\end{cases} \tag{*}
\]

This follows from \[\text{Lemma 1.1}\] It is useful to observe that, for all \( x, y \in S \) such that \( x \leq y \), if \( x \in S_2 \) then \( y \in S_2 \); if \( y \in S_1 \) then \( x \in S_1 \). When \((\dagger\dagger)\) is satisfied and \( D \neq \emptyset \) we refer to the poset \((S, \leq)\) as the \textit{gluing} of \((S_1, \leq_1)\) and \((S_2, \leq_2)\) along \( D \). Of course the relation \( \leq \) as defined in \((*)\) would not always be an ordering, if \((\dagger\dagger)\) were not assumed. For instance, we might have \( S_1 = \{a, d\} \), with \( d \leq_1 a \), and \( S_2 = \{b, d\} \), with \( b \leq_2 d \) (where \( a \neq b \neq d \neq a \)); in this case \( \leq \) is not transitive.

We are interested in gluing of lattices. As we are going to show, in this case the condition in \((\dagger\dagger)\) implies that \( D \) is a sublattice of both gluing lattices. We use \( \lor_1, \land_1, \lor_2 \) and \( \land_2 \) to denote the join and meet operations in the lattices \((S_1, \leq_1)\) and \((S_2, \leq_2)\), and use \( \inf_\leq \) and \( \sup_\leq \) with reference to the poset \((S, \leq)\).  

\[\text{Lemma 1.2.} \text{ Assume that } (S_1, \leq_1) \text{ and } (S_2, \leq_2) \text{ are lattices and } D \neq \emptyset. \text{ If } (\dagger\dagger) \text{ holds, then:} \]

\begin{enumerate}
\item for all \( a, b \in D \), \( a \lor_1 b = a \lor_2 b = \sup_\leq \{a, b\} \in D \);
\item for all \( a, b \in D \), \( a \land_1 b = a \land_2 b = \inf_\leq \{a, b\} \in D \);
\item \( D \) is a sublattice of both \((S_1, \leq_1)\) and \((S_2, \leq_2)\);
\item if \( i \in \{1, 2\} \), for all \( a, b \in S_i \), \( a \lor_i b = \sup_\leq \{a, b\} \) and \( a \land_i b = \inf_\leq \{a, b\} \). If \((S, \leq)\) is a lattice then \((S_i, \leq_i)\) is a sublattice of its.
\end{enumerate}
Proof. Let \( a, b \in D \) and let \( c = a \lor 1 \ b \). By the upward-closure of \( D \) in \( S_1 \), \( c \in D \).

Also, our assumptions give \( a \leq 1 \ c \) and \( b \leq 1 \ c \), hence \( c' := a \lor 1 \ b \leq 1 \ c \). But \( D \) is downward-closed in \( S_2 \), hence \( c' \in D \). Therefore \( c' \leq 1 \ c \) and so \( c' = c \). Finally, if \( y \in S \) and \( a, b \leq 1 \ y \) then \( y \in S_2 \) by an earlier remark, hence \( a, b \leq 1 \ y \) by Lemma 1.1 and \( c = a \lor 1 \ b \leq 1 \ y \). Thus \( c \) also is the supremum of \( \{a, b\} \) in \( (S, \leq) \).

Now (i) is proved. The proof of (ii) is dual, (iii) is an immediate consequence.

Now let \( a, b \in S_1 \). For all \( x \in S \) we have \( x \leq 1 \ a, b \) if and only if \( x \in S_1 \) and \( x \leq 1 \ a, b \), hence \( a \land 1 \ b = \inf_{\leq} \{a, b\} \). Let \( s = a \lor 1 \ b \), and let \( y \in S \) be such that \( a, b \leq 1 \ y \). If \( y \in S_1 \) then \( s \leq 1 \ y \), hence \( s \leq 1 \ y \). Otherwise, there exist \( d_a, d_b \in D \) such that \( a \leq 1 \ d_a \leq 1 \ y \) and \( b \leq 1 \ d_b \leq 1 \ y \). Then \( a, b \leq 1 \ d_a \lor 1 \ d_b = d_a \lor 1 \ d_b \leq 1 \ y \).

It follows that \( s \leq 1 \ d_a \lor 1 \ d_b \leq 1 \ y \), hence \( s \leq 1 \ y \). Therefore \( s = \sup_{\leq} \{a, b\} \). This proves (iv) for \( i = 1 \); the case \( i = 2 \) is proved similarly.

Thus, in the situation of the previous lemma \( D \) is a filter in \( S_1 \) and an ideal in \( S_2 \), which amounts to saying that in the case of lattices our notion of gluing coincides with that of gluing of lattices as defined, for instance, in [1], Section IV.2. However, it is worth noting that, contrasting with what is stated in [1], Lemma 298, it may well happen that the gluing \((S, \leq)\) is not a lattice:

**Example 1.3.** Consider the direct product of lattices \( \mathbb{Z} \times \{0, 1, 2, 3\} \), where the orderings in both components are the usual ones. Let \( x \) and \( y \) be two different elements, both outside \( \mathbb{Z} \times \{0, 1, 2, 3\} \). We let \( S_1 = (\mathbb{Z} \times \{0, 1, 2\}) \cup \{x\} \), and define an ordering \( \leq_1 \) on \( S_1 \) by extending the already defined ordering on \( \mathbb{Z} \times \{0, 1, 2\} \) and letting \( (n, 0) \leq_1 (n, 2) \) for all \( n \in \mathbb{Z} \), while no element of \( \mathbb{Z} \times \{1\} \) is \( \leq_1 \)-comparable with \( x \). This makes \((S_1, \leq_1)\) into a lattice. Similarly, the lattice \((S_2, \leq_2)\) is defined by letting \( S_2 = (\mathbb{Z} \times \{1, 2, 3\}) \cup \{y\} \), where \( \leq_2 \) extends the ordering on \( \mathbb{Z} \times \{1, 2, 3\} \), no element of \( \mathbb{Z} \times \{2\} \) is comparable with \( y \) and \( (n, 1) \leq_2 y \leq_2 (n, 3) \) for all \( n \in \mathbb{Z} \). Then \( D = \mathbb{Z} \times \{1, 2\} \) satisfies the closure conditions in \((\uparrow \downarrow)\), but it is easily shown that the set of all upper bounds of \( \{x, y\} \) in \((S, \leq)\) is \( \mathbb{Z} \times \{3\} \), hence \( \sup_{\leq} \{x, y\} \) does not exist. Thus \((S, \leq)\) is not a lattice.

The reason why the poset just discussed is not a lattice is the lack of completeness in \( D \). In fact we have:

**Proposition 1.4.** Assume that \((S_1, \leq_1)\) and \((S_2, \leq_2)\) are lattices and \((\uparrow \downarrow)\) is satisfied. If \( D \) is complete as a lattice, then \((S, \leq)\) is a lattice.

Proof. Let \( a, b \in S \). We know from Lemma 1.2 that both \( \sup_{\leq} \{a, b\} \) and \( \inf_{\leq} \{a, b\} \) exist if \( a, b \in S_1 \) or \( a, b \in S_2 \). So, to prove that \((S, \leq)\) is a lattice we only have to deal with the case where \( a \in S_1 \setminus S_2 \) and \( b \in S_2 \setminus S_1 \). Let \( D_a = \{d \in D \mid a \leq d\} \). Since \( D \) is complete \( D_a \) has a greatest lower bound \( d_a \) in \( D \) (with respect to \( \leq_1 \)). As \( D \) is upward-closed in \( S_1 \), \( t := a \lor 1 \ d_a \in D \), but \( t \leq_1 d \) for all \( d \in D_a \). Hence \( t = d_a \), that is, \( a \leq_1 d_a \). Let \( s := d_a \lor 2 \ b \); then
complete lattices is a lattice $a \leq_1 d_a \leq_2 s$, hence $a, b \leq s$. Now let $y$ be any upper bound of \{a, b\} in $S$. Then $b \leq y$, hence $b \leq_2 y$, and there exists $d \in D$ such that $a \leq_1 d \leq_2 y$. But then $d_a \leq_1 d$, so $s = d_a \lor_2 b \leq_2 d \lor_2 b \leq_2 y$. This shows that $s = \sup_\leq \{a, b\}$. A dual argument proves the existence of $\inf_\leq \{a, b\}$. Therefore $(S, \leq)$ is a lattice. 

When $D$ is finite the completeness condition on $D$ in Proposition 1.4 is trivially satisfied, as long as $D \neq \emptyset$. Therefore a gluing of two finite lattices is a lattice. More generally the next corollary shows that any gluing of two complete lattices is a lattice.

**Corollary 1.5.** Assume that $(S_1, \leq_1)$ and $(S_2, \leq_2)$ are complete lattices, $D \neq \emptyset$ and $(\uparrow \downarrow)$ is satisfied. Then also $D$ and $(S, \leq)$ are complete lattices.

**Proof.** Let $\emptyset \neq X \subseteq D$. Then $X$ has a least upper bound $a$ in $(S_1, \leq_1)$. By the upward-closure of $D$ in $S_1$, $a \in D$, hence $a$ is the least upper bound of $X$ in $D$. As a special case, this shows that $D$ has maximum, a dual argument shows that it also has minimum. It follows that $D$ is a complete lattice. By Proposition 1.4, $(S, \leq)$ is a lattice; we shall use $\lor$ and $\land$ for the join and meet operation in this lattice. Now let $X \subseteq S$. If $X \subseteq S_1$ then $X$ has a least upper bound $u$ in $S_1$. If $y$ is an upper bound of $X$ in $S$, then $y' := u \land y \in S_1$, and of course $y'$ is an upper bound of $X$, so $u \leq_1 y' \leq y$. Thus $u = \sup_\leq X$. Now assume $X \bot S_1$ and let $Y$ be the set of all upper bounds of $X$ in $S$. Then $Y \subseteq S_2$, hence, by dualizing the previous part of the argument, $Y$ has a greatest lower bound $v$ in $S$. Of course $v \in Y$, hence $v = \sup_\leq X$. Thus every subset of $S$ has a least upper bound in $(S, \leq)$, so this poset is complete. 

It can be noted that $(S, \leq)$ can be a lattice, even a complete one, when $D$ is not complete. For instance, the (complete) chain $Z \cup \{-\infty, +\infty\}$ of the integers with a minimum and a maximum element added can be realized as the gluing of two chains $Z \cup \{-\infty\}$ and $Z \cup \{+\infty\}$ along the incomplete chain $Z$.

It is also useful to observe that, still in the case when $(S_1, \leq_1)$ and $(S_2, \leq_2)$ are lattices and $D$ is complete the $(\uparrow \downarrow)$ condition means that $D$ must be the interval $[a/b]$ in the lattices $(S_1, \leq_1)$ and $(S_2, \leq_2)$, where $a = \max S_1$ and $b = \min S_2$. Therefore finite lattices can only be glued along intervals containing the greatest element of the first and the least elements of the second lattice.

We conclude this section with a few examples of how our definition of gluing of lattices can be visually rendered by Hasse diagrams. The easiest example is that of chains: gluing two (finite) chains can only produce a chain. Following standard notation, for any integer $n > 1$ we call $M_n$ the $(n + 2)$-element lattice consisting of a minimum, a maximum and $n$ atoms:
It is easy to describe all possible gluing of two lattices of this kind. Let $L_1 \simeq M_n$ and $L_2 \simeq M_m$, for integer $n, m > 1$. To glue $L_1$ and $L_2$ we must assume that their intersection is an interval $D = [a/b]$, (on which $L_1$ and $L_2$ induce the same order), where $a = \max L_1$ and $b = \min L_2$. It might be the case that one of the two lattices coincides with $D$, in which case the gluing will be the other lattice. Excluding this case, there are two possibilities up to isomorphism: either $a = b$ or $|D| = 2$, giving rise to two different types of lattices:

\[\begin{array}{c}
\begin{array}{c}
\ldots \\
\ldots
\end{array}
\end{array}\quad \begin{array}{c}
\begin{array}{c}
a
\ldots \\
\ldots
\end{array}
\end{array}\]

\[\begin{array}{c}
\begin{array}{c}
\ldots \\
b
\end{array}
\end{array}\]

2 Subgroup lattices

As an application of the discussion in the previous section, we consider, in a few small cases, the following problem: when is a gluing of two finite lattices $L_1$ and $L_2$ isomorphic to the lattice $\mathcal{L}(G)$ of all subgroups of some group $G$? (To avoid trivial cases we only consider proper gluing, that is, assume that neither of $L_1$ and $L_2$ is contained in the other one). In this case it is clear that $L_1 \simeq \mathcal{L}(H)$ for some $H \leq G$, so $L_1$ is a full subgroup lattice itself. $L_2$ might not satisfy this condition but it must at least satisfy the weaker requirement of being isomorphic to an interval in a subgroup lattice; so both $L_1$ and $L_2$ are complete. Note that, by [Corollary 1.5] any gluing of subgroup lattices is a complete lattice, even though not always a subgroup lattice (see below).

As a first instance of our problem, consider finite chains. As is well-known, a chain of finite length $n$ is isomorphic to the subgroup lattice of a cyclic group of order $p^n$ for a prime $p$, and only this case may occur. More generally:

**Proposition 2.1.** The subgroup lattice of a finite group $G$ is isomorphic to a proper gluing of two lattices $L_1$ and $L_2$, one of which is a chain, if and only if $G$ is either a cyclic $p$-group of order at least $p^3$ for some prime $p$ or isomorphic to the generalized quaternion group of order $2^n$ for some integer $n > 2$. In the former case both $L_1$ and $L_2$ are chains, in the latter $L_1$ is a chain but $L_2$ is not, being isomorphic to the subgroup lattice of the dihedral group of order $2^{n-1}$.

**Proof.** Suppose that $\mathcal{L}(G)$ is isomorphic to a proper gluing of lattices $L_1$ and $L_2$. If $L_2$ is a chain then it is clear that $G$ has only one maximal subgroup. Then $G$ is a cyclic $p$-group for some prime $p$, and it also follows that $L_1$ is a chain. Now suppose that $L_2$ is not a chain, hence $L_1$ is a chain. Then $G$ has only one (nontrivial) minimal subgroup but is not a cyclic group of prime-power order. Thus (see [2], 5.3.6) $G$ is a generalized quaternion 2-group. Then $Z := Z(G)$ is
the only subgroup of order 2 in $G$, and $G$ has more than two subgroups in which $Z$ is maximal. At least two of them, say $A$ and $B$, do not belong to the chain in $\mathcal{L}(G)$ corresponding to $L_1$, hence $A, B \in L_2$ and, therefore, $Z = A \cap B \in L_2$. It follows that $L_2 = \mathcal{L}(G) \setminus \{1\} \simeq \mathcal{L}(G/Z)$. But $G/Z$ is dihedral, hence $L_2$ is of the required type. The converse is obvious: if $G$ is a generalized quaternion group of order $2^n$ and $H$ is any nontrivial cyclic subgroup of $G$ then $\mathcal{L}(G)$ is a proper gluing of $\mathcal{L}(H)$ by a lattice isomorphic to the subgroup lattice of $G/Z(G)$, a dihedral group of order $2^n-1$. So the proof is complete. Note that any chain of length between 1 and $n - 1$ may occur as $L_1$ in the case just considered. \hfill $\square$

We remark that in the infinite case more complex examples can occur. In fact, the subgroup lattice of an extended Tarski group (see [3], page 82) can be realized as a proper gluing of a chain and $M_\infty$, the countable lattice in which all maximal chains have length 2.

Next we consider the case of iterated gluing of the lattices of type $M_n$ described in the previous section. We start by listing the cases in which the subgroup lattice of a group is isomorphic to one of these lattices. This result is well-known and can be read off the classification of finite modular groups (see for instance [3], Theorem 2.4.4).

**Lemma 2.2.** Let $n$ be an integer greater than 1. Then there exists a group $G$ such that $\mathcal{L}(G) \simeq M_n$ if and only if $n = 2$ or $n = p + 1$ for some prime $p$. More precisely, for a group $G$ we have:

(i) $\mathcal{L}(G) \simeq M_2$ if and only if $G$ is cyclic of order the product of two distinct primes;

(ii) $\mathcal{L}(G) \simeq M_{p+1}$ for some prime $p$ if and only if either $G$ is elementary abelian of order $p^2$ or is a nonabelian split extension $\langle x \rangle \times \langle y \rangle$, where $\langle x \rangle$ has order $p$ and the order of $y$ is a prime divisor of $p - 1$.

We say that the lattice $L$ is an *iterated gluing* of the (finite) sequence $(L_1, L_2, \ldots, L_t)$ of lattices to mean that $L$ is the union of the chain of sublattices $S_i = L_1 \cup L_2 \cup \cdots \cup L_i$, with $i$ ranging in $\{1, 2, \ldots, t\}$, in such a way that, for all subscripts $i < t$, $S_{i+1}$ is a gluing of $S_i$ and $L_{i+1}$. If this gluing is proper for all $i$, then we say that $L$ is a proper iterated gluing of the sequence; of course when dealing with iterated gluing only this case really needs to be taken into account.

The following piece of terminology will also be handy: we say that an element $y$ of a lattice $(L, \leq)$ is *one-headed* if the set $\{x \in L \mid x \leq y\}$ has maximum.

Extending some remarks in the previous section, we note that if a lattice $S$ is properly glued with one lattice $M \simeq M_n$ for some $n > 1$, then $S \cap M = \{a, b\}$, where $a = \max S$ and $b = \min M$, and either $a = b$ or $a$ is an atom of $M$. In either case, if $y$ is an atom of $M$ different from $a$ and $x$ is an element of the gluing $S \cup M$ such that $x < y$, then $x \in S$ and $x \leq b$; this follows from $\text{[Lemma 1.1]}$. Therefore $y$ is one-headed. A further consequence is that the co-atoms of $S \cup M$ are precisely the atoms of $M$. 

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These remarks make easy to describe the iterated gluing that we are dealing with. Let $L$ be a proper iterated gluing of the sequence $(L_1, L_2, \ldots, L_t)$ of lattices, where, for all subscripts $i$, there is an integer $n_i > 1$ such that $L_i \simeq \mathcal{M}_{n_i}$; we also let $a_i = \max L_i$ and $b_i = \min L_i$. It is clear that $a_i = \max S_i$ for all $i$, where $S_i$, as above, is the sublattice $L_1 \cup L_2 \cup \cdots \cup L_i$ of $L$. If $i < t$ we have $L_{i+1} \cap S_i = \{a_i, b_{i+1}\}$, and either $a_i = b_{i+1}$ or $a_i$ is a co-atom of $S_i \cup L_{i+1}$.

By the previous remarks (and an extra argument for the elements $b_i$), every element of $L$ which is not one of the $a_i$ nor $L = b_1$ is one-headed. Then $A := \{b_1\} \cup \{a_i \mid 1 \leq i \leq t\}$ is the set of all elements of $L$ which are not one-headed. But $A$ is a chain; it follows that every automorphism of the lattice $L$ fixes $A$ elementwise. Also, for all subscripts $i$, $b_i$ is the meet of all elements of $L$ covered by $a_i$, and so also the elements $b_i$ are fixed by all automorphisms of $L$.

We make a distinction that is very useful to our purposes. We say that our (proper, iterated) gluing $L$ is orthodox if and only if, for all $i$ such that $1 \leq i \leq t - 1$ we have $a_i \neq b_{i+1}$ (that is, $|L_i \cap L_{i+1}| = 2$) and, if $i \leq t - 2$, we also have $a_i \neq b_{i+2}$, or, equivalently, $L_i \cap L_{i+2} = \emptyset$. Here are a few examples of possible gluings in the case $t = 3$; only in the right-most case the gluing is orthodox:

It should be clear that, if the (proper) gluing $L$ as described above is assumed to be orthodox, then $L$ is determined, up to isomorphism, by the integer sequence $(n_1, n_2, \ldots, n_t)$. So, for any such sequence of integers (assumed to be greater than 1) we write $\mathcal{M}_{n_1 \cdot n_2 \cdot \ldots \cdot n_t}$ for an orthodox proper gluing of a sequence of lattices isomorphic to $\mathcal{M}_{n_1}, \mathcal{M}_{n_2}, \ldots, \mathcal{M}_{n_t}$, in that order.

We now state and prove the main result of this section:

**Theorem 2.3.** Let $t$ be a positive integer and let $L$ be a proper iterated gluing of a sequence $(L_1, L_2, \ldots, L_t)$ of lattices, where, for all $i \in I := \{1, 2, \ldots, t\}$, there exists an integer $n_i > 1$ such that $L_i \simeq \mathcal{M}_{n_i}$. Then there exists a group $G$ such that $\mathcal{L}(G) \simeq L$ if and only if $L \simeq \mathcal{M}_{n_1 \cdot n_2 \cdot \ldots \cdot n_t}$ and either $n_i = 2$ for all $i \in I$ or, for some prime $p$, either $n_i = p + 1$ for all $i \in I$, or $n_t = p + 1$ and $n_i = 2$ for all $i \in I \setminus \{t\}$.

If these conditions on the integers $n_i$ are satisfied and $G$ is a group, then $\mathcal{L}(G) \simeq \mathcal{M}_{n_1 \cdot n_2 \cdot \ldots \cdot n_t}$ if and only if one of the following holds:

(i) $G$ is cyclic of order $pq^t$ for some distinct primes $p$ and $q$. In this case $n_i = 2$ for all $i \in I$;
(ii) \( G = \langle x \rangle \times \langle y \rangle \), where \( x \) has prime order \( p \), \( y \) has order \( q^t \) for some prime \( q \), and \( y \) acts on \( \langle x \rangle \) by means of an automorphism of order \( q \) (so \([x, y^q] = 1\)). In this case \( q \) divides \( p - 1 \), \( n_t = p + 1 \) and \( n_i = 2 \) for all \( i \in I \setminus \{t\} \);

(iii) \( G \) is the direct product of a group of prime order \( p \) by a cyclic one of order \( p^t \). In this case \( n_i = p + 1 \) for all \( i \in I \);

(iv) \( G = \langle x \rangle \times \langle y \rangle \), where \( t > 1 \) and, for some prime \( p \) such that \( p \neq 2 \) if \( t = 2 \), \( x \) has order \( p^t \), \( y \) has order \( p \) and \( x^y = x^{1+p^{t-1}} \). Also in this case \( n_i = p + 1 \) for all \( i \in I \).

**Proof.** We may assume \( t > 1 \). Let \( \theta : L \to \mathcal{L}(G) \) be a lattice isomorphism. As in the preceding discussion, let \( a_i = \max L_i \) and \( b_i = \min L_i \) for all \( i \in I \).

Our first aim is showing that \( L \) is an orthodox gluing. If \( y \) is a one-headed element of \( L \) then \( y^g \) is a subgroup of \( G \) with only one maximal subgroup. Therefore \( y^g \) is cyclic of prime-power order and \( \mathcal{L}(y^g) \) is a chain. It follows that \( L_{\leq y} := \{ x \in L \mid x \leq y \} = (\mathcal{L}(y^g))^{\theta-1} \) is a chain. Now fix a subscript \( i < t \). As we observed, if \( y \) is an atom of \( L_{i+1} \) and \( y \neq a_i \) then \( y \) is one-headed. Hence, \( L_{\leq y} \) (as just defined) is a chain. If \( a_i = b_{i+1} \) then \( L_i \subseteq L_{\leq y} \), and this is a contradiction. We obtain a similar contradiction if \( i < t - 1 \) and \( a_{i-1} = b_{i+1} \).

This shows that \( L \) is orthodox, as required. Therefore \( \mathcal{L}(G) \cong L \cong M_{n_1n_2...n_t} \).

For all \( i \in I \), let \( A_i = a_i^\theta \) and \( B_i = b_i^\theta \). Since the elements \( a_i \) and \( b_i \) are fixed by all automorphisms of \( L \), the subgroups \( A_i \) and \( B_i \) are characteristic in \( G \).

Now let \( y \) be a co-atom of \( L \) (that is, an atom of \( L_t \)) different from \( a_{t-1} \). Again, \( y \) is one-headed, hence \( Q := y^g \) is a cyclic \( q \)-group for some prime \( q \). For all \( i \in I \setminus \{t\} \), \( b_{i+1} \) is an atom of \( L_i \), then \( b_1 < b_2 < \cdots < b_{i-1} < y < a_i \) is a maximal chain in \( L \). Therefore \( 1 = B_1 < B_2 < \cdots < B_t < Q \) is a composition series of \( Q \). So \( |Q| = q^t \). Let \( P \) be one of the minimal subgroups of \( G \) different from \( B_2 \) (that is, \( P = x^g \) for some atom \( x \) of \( L \) different from \( b_2 \)); then \( p := |P| \) is prime. For all \( i \in I \setminus \{t\} \), \( a_{i-1} \) covers \( b_i \), hence \( B_i \) is a maximal subgroup of \( A_{i-1} \); it follows that \( A_{i-1} = B_i \ltimes P \). Also, \( G = A_t \downarrow Q \), hence \( G = A_t = PQ \) and \(|G| = pq^t\).

Assume \( p \neq q \). Then \( P \) is a Sylow \( p \)-subgroup of \( A_1 = B_2 \ltimes P \), and the Frattini argument (see, for instance, [2], 5.2.14) shows \( G = A_1 \mathcal{N}_G(P) \). It follows that \( \mathcal{N}_G(P) \not\trianglelefteq A_{t-1} \). Thus \((\mathcal{N}_G(P))^{\theta-1} \not\leq L_1 \cup L_2 \cup \cdots \cup L_{t-1} \) and so \((\mathcal{N}_G(P))^{\theta-1} \in L_t \).

Therefore \( B_t \leq \mathcal{N}_G(P) \). Hence \( A_{t-1} = P \times B_t \) is cyclic. On the one hand this shows that \( n_i = 2 \) for all \( i \neq t \); on the other hand it implies that \( P \not\trianglelefteq G \), hence \( G = P \rtimes Q \). If \( Q \trianglelefteq G \) then \( G \) is cyclic and (i) holds. Otherwise, if \( Q \not\trianglelefteq G \), then \( G/B_t = (A_{t-1}/B_t) \rtimes (Q/B_t) \) is not abelian and \( L_t \simeq \mathcal{L}(G/B_t) \simeq M_{p+1} \) (see [Lemma 2.2]). Also, \( Q \) acts on \( P \) with order \( q \), because \( B_t = Q^q \). The automorphism group of \( P \) has order \( p - 1 \), hence \( q \) divides \( p - 1 \) and so (ii) holds.

Now assume \( p = q \), so that \( G \) is a \( p \)-group and hence \( Q \trianglelefteq G \). Thus \( G = P \rtimes Q \).

For all \( i \in I \) we have \( L_i \simeq \mathcal{L}(A_i/B_i) \), and \( A_i/B_i \) is elementary abelian, therefore \( n_i = p + 1 \). If \( P \not\trianglelefteq G \), that is, \( G = P \rtimes Q \) is abelian, we obtain (iii). Even if \( G \) is not abelian we have \( A_{t-1} = P \rtimes B_t = P \rtimes Q^p \). So, if we let \( Q = \langle x \rangle \) and \( P = \langle y \rangle \) then \( x^y = x^r \) for some integer \( r \equiv_{p-1} 1 \). At the expense of replacing \( y \) with
a different generator of \( P \), if needed, we may assume \( r = 1 + p^{t-1} \). If \( p = 2 = t \) then \( G \) would be isomorphic to \( D_8 \), the dihedral of order 8. This is a impossible, because \( D_8 \) has five subgroups of order 2. So this case is excluded, and (iv) holds.

To complete the proof we have to show that, conversely, if \( G \) is a group as described in one of (i–iv) then \( \mathcal{L}(G) \) has the required structure. This is obvious in cases (i) and (iii), when \( G \) is abelian. Assume (ii), and let \( A = \langle x, y^\theta \rangle \). Then \( A = \langle x \rangle \times \langle y^\theta \rangle \) and so \( \mathcal{L}(A) \) is isomorphic to the orthodox proper gluing of \( t-1 \) copies of \( M_2 \). Let \( L = \left[ G/\langle y^\theta \rangle \right] \), the interval \( \{ H \leq G \mid y^\theta \in H \} \) of \( \mathcal{L}(G) \). Since \( \langle y^\theta \rangle = Z(G) \leq G \) and \( G/Z(G) \) is not abelian, we have \( L \simeq \mathcal{L}(G/\langle y^\theta \rangle) \simeq M_{p+1} \).

So, it will be enough to check that \( \mathcal{L}(G) \) is a gluing (of the required kind) of \( \mathcal{L}(A) \) and \( L \). Clearly \( \mathcal{L}(A) 
 L = \{ A, \langle y^\theta \rangle \} \), and \( A = \max \mathcal{L}(A) \neq \min L = \langle y^\theta \rangle \), so, all we have to prove is that \( \mathcal{L}(G) = \mathcal{L}(A) \cup L \), that is, for all \( H \leq G \) either \( H \leq A \) or \( \langle y^\theta \rangle \leq H \). Let \( H \leq G \). Let \( S_p \) and \( S_q \) be Sylow \( p \)- and \( q \)-subgroups of \( H \), respectively. Then \( S_p \) is either 1 or \( \langle x \rangle \), because \( \langle x \rangle \) is the only Sylow \( p \)-subgroup of \( G \). On the other hand, \( S_q \) is contained in a conjugate \( Y \) of \( \langle y^\theta \rangle \), hence either \( S_q = Y \) or \( S_q \leq Y^\theta = Z(G) = \langle y^\theta \rangle \). In the former case \( H \geq S_q \geq \langle y^\theta \rangle \); in the latter \( H = S_pS_q \leq \langle x \rangle \langle y^\theta \rangle = A \). So \( \mathcal{L}(G) = \mathcal{L}(A) \cup L \), as claimed. Finally, assume that \( G \) is of the type described in (iv). By a similar argument as in the previous case, we only need to show that all subgroups of \( G \) which are not contained in \( A := \langle x^p, y \rangle \) contain \( \langle x^p \rangle \). Let \( H \leq G \). If \( \langle x \rangle \neq H \leq A \) then \( H \) contains \( yx^\lambda \), for some integer \( \lambda \) not divisible by \( p \). Direct calculation, using the fact that \( G' \) is central and has order \( p \), and \( y^p = 1 \), yields \( (yx^\lambda)^p = x^{p\lambda} \) if \( p \neq 2 \) and \( (yx^\lambda)^2 = x^{(2^t+2^{t-1})\lambda} \) if \( p = 2 \). In the latter case we also have \( t > 2 \), so, in either case, \( \langle yx^\lambda \rangle^p = \langle x^p \rangle \); thus \( x^p \in H \). So the required conclusion follows, and the proof is complete.  

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References


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