A Geometrical Solution by Fermat to

a Problem of Maximum

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Abstract

In this paper the method devised by Fermat for the determination of the maximum and minimum of a function is exposed together with the method for the tangents. Moreover a problem of maximum geometrically solved by Fermat is studied.

Keywords: Fermat, maximum, minimum, tangent, subtangent

1. Introduction

In the sequel the method elaborated by Pierre Fermat (1601-1665) to determine the minimum and maximum value of a function in his paper “Methodus ad disquirendam maximam et minimum” (written around 1629 but published in 1679 only, after his death, by his son Samuel) will be exposed, and in particular a problem of maximum solved by him geometrically without using such a method will be commented on.

How well known Fermat’s method, that can be viewed as the birth of the infinitesimal calculus, was censured by many of the subsequent mathematicians and philosophers: all the criticism finished only in 1821 with the publication of the Course de Analyse by Cauchy (see [3]), where the concepts of limit of a function and derivative were rigorously defined for the first time. Indeed, as we will see, Fermat’s method violates in a showy way the principle of not contradiction (it cannot be A and not A), that with the principle of identity (A is A)
and the principle of the third excluded (either A or not A) portrays the basis of the Aristotelian logic. The Aristotelian logic and philosophy had prevailed in the philosophy schools during all the Middle Age and in the preceding centuries, representing the unquestionable truth of every knowledge, both in the political and in the religious field.

During the sixteenth century there had been deep changes in society, great innovations had been introduced in astronomy (it is enough to remember Copernicus and Kepler), in geography (the American explorations), in the religious field (the Reformation and the Counter-Reformation).

In the first half of the seventeenth century, particularly in France, the classic culture, the Aristotelian logic and philosophy met with the strong opposition of many philosophers; in 1624 a group of philosophers in Paris decided to discuss in public 14 theses against Aristotle: immediately the French Parliament enjoined them not to attack Aristotle and the classics under pain of death.

Also Descartes wrote several passages against the Aristotelian logic, since he considered it not suitable in order to discover new truths but only helpful to arrange what is already known. In the Second Part of his “Discours de la méthode” (1637) he writes: “With regard to the logic, its syllogisms and the widest part of its principles are more useful for the communication of what we already know than to investigate what is unknown”. This passage gives an almost perfect description of what was happening in the mathematical framework: indeed the Fermat’s rejection of the principle of not contradiction, which only in the cultural climate described above could come true, is united with the enthusiasm for the large amount of new results that he is able to achieve by his method and the easiness with which he succeeds in it.

2. Fermat’s method for the minimum and maximum

First of all it is necessary to remember that, anticipating the Geometry by Descartes (see [7]), Fermat had already exposed the analytic geometry (see [2]) in a clear manner, which is preferable to Descartes exposition from a didactical point of view, individualizing the points of the plane by two numbers, which nowadays we call Cartesian coordinates and describing straight lines and curves by means of their equations: this is done in the Ad locos planos et solidos isagoge, a paper he begun to write in 1629 but was only published in 1679, after his death (see[4]). Around 1629 he wrote also his Methodus ad disquirendam maximam et
A geometrical solution by Fermat to a problem of maximum, based on the following consideration: when a function $f$ reaches its maximum or minimum value corresponding to a point $x$, the values $f$ assumes in the points $x+E$ near to $x$ are very near to the value $f(x)$, so near that, even if the difference $f(x+E)-f(x)$ is divided by $E$, the ratio is very near to 0, that is
\[
\frac{f(x+E)-f(x)}{E} \approx 0;
\]
here the symbol $\cong$ does not denote an equality, but, with Fermat’s terminology, an “adequality”, that is an approximate equality. Then Fermat’s method consists in simplifying the ratio on the left of the preceding adequality (what is always possible if the given function is rational or irrational) and in equalizing it to 0, putting at the same time $E = 0$.
Fermat furnishes many examples in which the solution is already well known in order to prove the effectiveness of his method.

**Example** – Given the segment $AC$, determine a point $B$ on it in such a way that the rectangle whose edges are $AB$ and $BC$ has largest area (see [4], page 134).

**Resolution** – Let $a$ be the length of $AC$ and let $x$ be the length of $AB$. Then the function that has to be maximized is $f(x) = x(a-x)$. Consider the adequality:
\[
\frac{f(x+E)-f(x)}{E} = -2x+a-E \cong 0.
\]
Then we obtain, putting $E=0$, the resulting equation $-2x+a = 0$, whence $x = \frac{a}{2}$, that is, among all the rectangles with given perimeter $2a$, the square has the largest area.

Observe that, even if the principle of not contradiction is evidently violated, from an intuitive point of view the reasoning looks justified, since $E$ is not to be considered fixed once and for all, but is variable and the result can be considered as the consequence of an approximation procedure obtained giving $E$ smaller and smaller values. Obviously in such a way only an approximate value can be obtained as a result, while Fermat passes, putting $E=0$, from an adequality to an equality: this is the problem and the cause of the following criticism.
Fermat in *IV. Methodus de maxima et minima* (see[4] page 148) considers again the preceding example however exposing another method he has devised but he does not use in general since he thinks it is more complicated. It is based on the following observation: near a point $x$ of maximum or minimum there are infinite many couples of points where the function assumes the same values; therefore if $x_1$ and $x_2$ is such a couple it is

$$f(x_1) = x_1(a - x_1) = f(x_2) = x_2(a - x_2).$$

By such an equality, dividing by $x_1 - x_2$ we obtain $a = x_1 + x_2$. At this point Fermat puts $x_1 = x_2 = x$ whence $a = 2x$, as before. Of course also for this method the same objections raised for the first method can be made.

**3. The method for the tangents**

Fermat traces the determination of the tangents to the curves (see [2], [5] and [6]) back to the method for the maximum and minimum: he uses a procedure we can explain by modern terminology in the following way: a function $f$ and a point $P = (x, f(x))$ are given. Let $Q = (x,0)$; consider the tangent to the graph at the point $P$ and let $T$ be the intersection point with the x-axis and $K$ the point on it whose abscissa is $x+E$. Let also $R = (x+E,0)$. Fermat is interested in the determination of the subtangent, that is the determination of the length of the segment $TQ$. To this end he considers the two similar triangles $TKR$ and $TPQ$: if $E$ is small enough the point $K$ can be considered coincident with the point $M = (x+E, f(x+E))$ and therefore it is
possible to write the proportion $MR : PQ = TR : TQ$ whence, passing to the lengths (observe that all the geometry of Fermat and also of Descartes is developed in the first quadrant), we have: $f(x+E) : f(x) = (c+E):c$. By applying the subtrahend, we obtain: 
\[
\frac{f(x+E)-f(x)}{E} \approx \frac{f(x)}{c}.
\]

Also this equality is of course an “adequality”: simplifying $E$ on the first side and putting then $E = 0$ it is possible, given $x$, to determine the value $c$. Obviously the same criticism for the maximum and minimum theory holds also for this method, but it is easy to apply and gives good results in a wide class of functions.

4. A maximum problem geometrically solved

As we have already said, Fermat suggests several applications of his method either for the research of the maximum and minimum value of a functions or for the determination of the tangents: many of these applications serve the purpose to prove in a well known framework how good the method is, but Fermat sometime solve some open question also, as when he proves for the first time in a rigorous way the refraction law (see [4] and [1]).

Among the many problems he solves there is one, exposed in an appendix (see [4] page 157), where however Fermat does not use his analytic method (that is applicable as we will see) but unexpectedly he solves by means of geometrical considerations, a more secure way, of course, and also, as he says, a more elegant way.

**Problem** – Consider the semicircumference $FBD$ whose diameter is $FD$; let $BH$ be the perpendicular on the diameter $FD$: find the maximum of the product $FH \cdot HB$. 

\[ \text{Diagram:} \]

\[ A \]

\[ F \]

\[ B \]

\[ M \]

\[ H \]

\[ D \]

\[ C \]
**Resolution by Fermat** - We will consider a Cartesian coordinate system with origin in $F$, $x$-axis on the diameter $FD$ and $y$-axis tangent to the semicircumference in $F$.

Fermat observes that the problem consists in determining, among all the hyperbolas whose equation is $xy = k$, the one that is tangent to the semicircumference: of course, if $B$ is the tangency point of the two curves, they have in $B$ the same straight line as tangent. By a proposition by Apollonius proved in the *Conics*, if $A$ and $C$ are the points of intersection of the tangent with the $y$-axis and the $x$-axis respectively, then $AB = BC$. Let $M$ be the center of the semicircumference and let $BN$ be the perpendicular from $B$ to the $y$-axis. Then the triangle $MBH$ is similar to the triangle $ANB$; moreover the hypotenuse $AB$ is equal to $AF$, since $AB$ and $AF$ are the tangents to the semicircumference drawn from $A$; therefore $AB$ is twice $AN$ (since the triangle $ABN$ is similar to the triangle $AFC$ and $AC$ is twice $AB$, by the Apollonius proposition), then also $BM$ is twice $MH$. Finally $FH = FM + MH$ is equal to $\frac{3}{2}$ of the radius of the semicircumference and the problem is solved.

**5. An analytic solution**

Now we will prove that the preceding problem can be easily solved by Fermat’s method exposed in Section 2. Indeed, let $x$ be the abscissa of $B$, $r$ the radius, then the ordinate $BH$ is given by $\sqrt{r^2 - (x - r)^2}$ and the function to maximize is

$$f(x) = \sqrt{r^2 - (x - r)^2}.$$ 

Consider two points $x_1$ and $x_2$ such that $f(x_1) = f(x_2)$, squaring the two sides we obtain:

$$x_1^2 (2r - x_1) = x_2^2 (2r - x_2).$$

Whence, after dividing for $x_1 - x_2$:

$$(2rx_1^2 + x_1x_2 + x_2^2) - (x_1^2 + x_2^2)(x_1 + x_2) = 0.$$

Finally put $x_1 = x_2 = x$: we thus obtain an equation whose solutions are $x = 0$, extreme of the interval of variability of $x$, corresponding to the minimum of the given function and $x = \frac{3}{2} r$, that is the required solution.
Observe that in the analytic solution of the Problem it is not necessary to use the hyperbolas and the proposition of Apollonius quoted above. However, of course, it can be proved very easily by the tangent method exposed in Section 3 that, given the hyperbola of equation $xy = k$, considered the tangent in the point B of abscissa $x$, if A and C are the points of intersection of the tangent with the y-axis and the x-axis respectively, then $AB = BC$. Indeed consider on the tangent to the hyperbola the point $B'$ whose abscissa is $x + E$; since the triangles $HBC$ and $KB'C$ are similar, mistaking the curve for the tangent, we can establish the “adequality”:

$$\frac{k}{x} : \frac{k}{x+E} \cong c : (c-E).$$

By applying the subtrahend we obtain:

$$\left(\frac{k}{x} - \frac{k}{x+E}\right) : \frac{k}{x} \cong E : c,$$

where $c$ is the subtangent. Simplifying and putting $E = 0$, we obtain $c = x$, that is $FH = HC$. Then, since the triangles $AFC$ and $BHC$ are similar it is also $AB = BC$ and the proposition is proven.

### 6. Conclusion

It is clear that Fermat prefers the geometric rather than the analytic method, but he thinks that, while the analytic method furnishes a mechanic and uniform procedure that works well for a wide class of problems, the geometric method requires an “ad hoc” solution for every type of problem, fantasy and inventive ability, rather different qualities from the skill necessary to learn a procedure and to apply it, always equal to itself, when it is the case.

Such a comparison can be very useful, in my opinion, from a didactic point of view, because it casts light upon two different ways to approach mathematics. Indeed it is possible to decide that it is satisfactory to improve the calculus ability of the students on the basis of some rules (for example this happens when they learn to execute algebraic expressions or learn to derive functions). But it is also possible to require the participation of their minds to a deeper level of liberty and fantasy. Of
course it is obvious that they have to learn to apply rules: but this is not the only purpose of a mathematical education.

It is very hard to learn how to build new rules; but perhaps this can be obtained solving easy problems departing from the traditional schemes: this can be viewed as a form of training for using what is already known in a new way, with the aim to prove some property. The preceding exercise can improve some students qualities that can be very useful either for their personal lives or in the social framework: indeed now it is not clear yet what will be required our students have to do when they leave the school or the university: what can be useful for that? Of course the aptitude for learning is important as the knowledge of the fundamental theoretical and practical aspects of their work, but it is not enough: the cleverness to solve new problems inventing new ways of solution is really essential, in my opinion.

**References**


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