On the Periodic Solutions of a Generalized SVEIR Model under Impulsive Vaccination

Raul Nistal, Manuel de la Sen, Santiago Alonso-Quesada

Department of Electricity and Electronics, Faculty of Science and Technology
University of the Basque Country, UPV/EHU
48940 Leioa, Spain

Asier Ibeas

Department of Telecommunications and Systems Engineering
Universitat Autonoma de Barcelona, UAB
08193 Barcelona, Spain

Abstract

A SVEIR model with delays is proposed with an impulsive vaccination applied at time instants regularly distributed. In this paper we present an analytic solution for the disease free periodic state derived from this model and demonstrate the uniqueness of the obtained solution. A more general model which shares this periodic solution is proposed.

Keywords: Epidemic Model, SVEIR, Vaccination

1 Introduction

The study of mathematical epidemiology contains hundreds if not thousands of epidemic compartmental models describing the spread of a disease in a population [1, 2, 3]. The reaction of the authorities and medical organizations in order to treat the disease and reduce new infections is usually described in these models with quarantine and vaccination [4, 5]. Given the limitations
in the production of vaccines, the infrastructure of their distribution or the biologic processes of the immune response, sometimes vaccination cannot be complete or sufficiently regular [6, 7]. The model presented in this paper contains an impulsive vaccination strategy which gives a resistance to the disease and eventually an immunity to a fraction of the susceptible subpopulation. The study of the dynamics of the subpopulations under this circumstances allows us to know whether the introduction of the disease in the population may result in an epidemic with a large number of cases or if the disease will extinguish by itself [8, 9, 10, 11, 12].

2 The SVEIR model

A generic SVEIR model of five subpopulations with two delays based on previous models [13, 14, 15] is proposed. The model considers the application for all time of a regular non-impulsive vaccination to a part of the arriving susceptible subpopulation as well as a regular impulsive vaccination to fraction of the whole susceptible subpopulation at instants uniformly distributed in time. Both transitions, from vaccinated and infectious subpopulations to the recovered one, lead to an immunity indistinguishable from each other. Natural death rate $b_2$ is the inverse of the life expectancy. In the same way, the transition rates $\gamma$ and $\gamma_1$ are the inverse of the average times individuals spend at the infectious subpopulation and the vaccinated one after becoming recovered.

Notation

Subpopulations

The subpopulations presented in the model are described by the following terms: $S(t)$ represents the subpopulation susceptible to the disease, $V(t)$ the subpopulation which has been vaccinated. The infected subpopulations are described by $E(t)$ and $I(t)$, the subpopulation exposed to the disease although not sick or infectious yet, and the subpopulation which fully develops the disease and is able to infect others, respectively. $R(t)$ is the subpopulation immune due to vaccination or being recovered from the disease, and the sum of all subpopulations is represented as $N(t)$.

Parameters

Observe that the values of all the parameters introduced here, involving the epidemic model, are non-negative. $b_1$, $b_3$ are the birth rates of the population, a constant one ($b_1$) and a population-dependent one ($b_3$). The natural death rate of any subpopulation is described by $b_2$. The constants $\gamma$, $\gamma_1$ correspond
to the ratio of transition to recovered from infected (I→ R) and vaccinated (V→ R) subpopulations, respectively, while the α describes the extra death rate caused by the disease in the infected (I) subpopulation. τ and ω are the average time of transition from exposed to infected (E→ I) and from immune to susceptible (R→ S) subpopulations respectively. β is the parameter associated to the disease transmission constant. η is the constant saturation related to the transmission of the disease which defines the incidence rate.

A diminishing factor δ is related to the disease transmission in the vaccinated subpopulation in contrast to that corresponding to the susceptible one. For the regular vaccination, the fraction of the population which is vaccinated since birth is \( V_c \in [0, 1] \), while the fraction of the susceptible subpopulation affected by the impulsive vaccination and the time intervals between two consecutive impulsive vaccination instants are described by θ and \( t_v \) respectively. The infectious incidence rate in the susceptible subpopulation is proportional to \( \beta \) and depends on \( I(t) \) and \( S(t) \). Due to the effects of the impulsive vaccination, there is a great variance in the value of the susceptible subpopulation, so a saturation factor is introduced in order to maintain a reasonable infection rate whether the value of \( S(t) \) is high or low. This saturation factor is of the form \( \frac{1}{1+\eta S(t)} \) with \( \eta \in \mathbb{R} \), \( \eta > 0 \). A similar incidence rate occurs in the vaccinated subpopulation with the parameter \( \beta \) reduced by a diminishing factor \( \delta \epsilon [0, 1] \), which implies the reduced possibility of a successful contagion of the disease in this subpopulation, and a saturation factor given by \( \frac{1}{1+\eta V(t)} \). The SVEIR model with delays is described by the following equations:

\[
\begin{align*}
\dot{S}(t) &= b_1 - b_2 S(t) - \frac{\beta S(t) I(t)}{1+\eta S(t)} + e^{-b_2 \omega} (\gamma I(t) - \omega) + \gamma_1 V(t - \omega) + b_3 (1 - V_c) N(t) \\
\dot{V}(t) &= -\theta \frac{\beta V(t) I(t)}{1+\eta V(t)} - \gamma_1 V(t) - b_2 V(t) + b_3 V_c N(t) \\
\dot{E}(t) &= \beta \left[ \frac{S(t) I(t)}{1+\eta S(t)} + \frac{\theta V(t) I(t)}{1+\eta V(t)} - e^{-b_2 \tau} \left( \frac{S(t-\tau) I(t-\tau)}{1+\eta S(t-\tau)} + \frac{\theta V(t-\tau) I(t-\tau)}{1+\eta V(t-\tau)} \right) \right] - b_2 E(t) \\
\dot{I}(t) &= \beta e^{-b_2 \tau} \left( \frac{S(t-\tau) I(t-\tau)}{1+\eta S(t-\tau)} + \frac{\theta V(t-\tau) I(t-\tau)}{1+\eta V(t-\tau)} \right) - (b_2 + \alpha + \gamma) I(t) \\
\dot{R}(t) &= \gamma_1 V(t) + \gamma I(t) - b_2 R(t) - (\gamma I(t - \omega) + \gamma_1 V(t - \omega)) e^{-b_2 \omega}
\end{align*}
\]

\[
\begin{align*}
S(t^+) &= (1 - \theta) S(t) \\
V(t^+) &= V(t) + \theta S(t) \\
E(t^+) &= E(t) \quad \text{if } t = nt_v \ (n = 1, 2, 3, \ldots) \\
I(t^+) &= I(t) \quad \theta \epsilon [0, 1] \\
R(t^+) &= R(t)
\end{align*}
\]
with \( N(t) = S(t) + V(t) + E(t) + I(t) + R(t) \).

Through the paper the notation for the left limit at the impulse time instants \( nt_v \) will be simply denoted by \( nt_v \). The parameters \( \omega \) and \( \tau \) are the internal delays at \((2.1),(2.5)\) and \((2.3),(2.4)\) respectively. The equation \((2.6)\) is an impulsive function representing a vaccination campaign acting periodically on a fraction \((0 \leq \theta \leq 1)\) of the susceptible subpopulation, which is converted into vaccinated subpopulation. Note that the presence of delays is often relevant in dynamic systems \([16, 17]\) and that the migrations from vaccinated and infectious to the susceptible subpopulation (through the temporary immune recovered subpopulation) are taken into account.

### 3 Disease-free equilibrium point with no impulsive vaccination

In order to study the equilibrium points, first we will consider the SVEIR model with regular non-impulsive vaccination, i.e., \( \theta = 0 \) in \((2.6)\), and a constant vaccination rate \( V_c \). Let \( S^*, V^*, E^*, I^*, R^* \) be the respective subpopulations at the eventual equilibrium points, i.e.: \( \lim_{t \to \infty} (S(t), V(t), E(t), I(t), R(t))^T = (S^*, V^*, E^*, I^*, R^*)^T \). Since the values of the subpopulations at an equilibrium point are constant, delay-dependency disappears at the equilibrium so that \( \lim_{t \to \infty} I(t - \tau) = \lim_{t \to \infty} I(t - \omega) = \lim_{t \to \infty} I(t) = I^* \) and \( \lim_{t \to \infty} E(t - \tau) = \lim_{t \to \infty} E(t) = E^* \). The model equations \((2.1)-(2.5)\) lead to:

\[
\begin{align*}
\frac{d}{dt} S^* &= -b_1 S^* + b_2 I^* + \beta S^* I^* + \gamma I^* + \gamma_1 V^* e^{-b_2 \omega} + b_3 (1 - V_c) N^* \\
0 &= \eta S^* - \gamma V^* - b_2 V^* + b_3 V_c N^* \\
(1 - e^{-b_2 \tau}) \beta \left( \frac{S^*}{1 + \eta S^*} + \frac{V^*}{1 + \eta V^*} \right) I^* - b_2 E^* = 0 \\
e^{-b_2 \tau} \beta \left( \frac{S^*}{1 + \eta S^*} + \frac{V^*}{1 + \eta V^*} \right) I^* - (b_2 + \alpha + \gamma) I^* = 0 \\
(1 - e^{-b_2 \omega}) (\gamma_1 V^* + \gamma I^*) - b_2 R^* = 0 \\
S^* + V^* + E^* + I^* + R^* &= b_1 - (b_2 - b_3) N^* - \alpha I^* = 0
\end{align*}
\]

for the purpose of obtaining the respective subpopulations at the equilibrium points. By assuming the condition of non-negativity for all subpopulations, i.e., \((S^*, V^*, E^*, I^*, R^*)^T \geq 0\), the solution of the equation \((3.1)\) reveals a set of points at which the equilibrium is reached. A solution of \((3.1)\) such that \( I^* \neq 0 \) is defined as an endemic equilibrium point and, if \( I^* = 0 \) then we say it is a disease-free equilibrium point. The model discussed here presents only one disease-free equilibrium (DFE) point, where \( I^* = 0 \) and \( E^* = 0 \). The values of the susceptible, vaccinated and recovered subpopulation as well as the total population at such a DFE point are obtained from the equations in \((3.1)\) by
introducing \( I^* = E^* = 0 \). In this way:

\[
\begin{align*}
N^* &= \frac{b_1}{b_2 - b_3} \\
S^* &= \frac{b_1}{b_2} \left[ 1 + \frac{b_3}{b_2 - b_3} \left( 1 + V_c \left( \frac{e^{-b_2 \omega}}{b_2 + \gamma_1} - 1 \right) \right) \right] \\
V^* &= \frac{V_c}{(\gamma_1 + b_2)(b_2 - b_3)} \\
R^* &= \frac{(1 - e^{-b_2 \omega})}{b_2} \gamma_1 V^* = V_c b_1 b_3 \gamma_1 (1 - e^{-b_2 \omega}) \\
&= \frac{b_1 b_3 \gamma_1}{b_2 (\gamma_1 + b_2)(b_2 - b_3)}
\end{align*}
\]

4. Regular impulsive vaccination around the disease free equilibrium point

The behavior of the model under a regular impulsive vaccination is studied in this section. The main motivation is to mitigate and, potentially, eradicate the infection from the host population when the DFE point is unstable under a regular non-impulsive vaccination strategy. The regular impulsive vaccination is characterized by a constant vaccination rate \( \theta \) and a constant inter-vaccination time interval \( t_v \). We will apply this vaccination strategy to an auxiliary model that we construct from the original model (2.1)-(2.6), in which there are neither exposed nor infectious subpopulations, \textit{i.e.}, \( E(t) = 0 \) and \( I(t) = 0 \ \forall t \geq t_0 \), being \( t_0 \) the hypothetical time instant at which the disease has been eradicated. The results we obtain in this auxiliary model would be analogous to our SVEIR model when it hypothetically tends to the disease-free regime. The dynamics equations of this reduced model \( \forall t \geq t_0 \) are:

\[
\begin{cases}
\dot{S}'(t) = b_1 - b_2 S'(t) + b_2 (1 - V_c) N'(t) + \gamma_1 V'(t - \omega) e^{-b_2 \omega} \\
\dot{V}'(t) = -\gamma_1 V'(t) - b_2 V'(t) + b_3 V_c N'(t) \\
\dot{R}'(t) = \gamma_1 (V'(t) - V'(t - \omega) e^{-b_2 \omega}) - b_2 R'(t)
\end{cases}
\] (4.1)

and

\[
\begin{cases}
S'(t^+) = (1 - \theta) S'(t) \\
V'(t^+) = V'(t) + \theta S'(t) \\
R'(t^+) = R'(t)
\end{cases}
\] (4.2)

The equation of the total population in such a disease-free situation is:

\[
\dot{N}'(t) = \dot{S}'(t) + \dot{V}'(t) + \dot{R}'(t) = b_1 - (b_2 - b_3) N'(t)
\] (4.3)

Such a total population presents a time evolution given by \( N'(t) = N^* - (N^* - N_0) e^{-(b_2 - b_3)(t - t_0)} \) where \( N_0 \) denotes the initial value \( N_0 = N'(t_0^*) \geq 0 \).
By supposing that \( b_2 > b_3 \), it follows that \( \lim_{t \to \infty} N'(t) = N^* \) and the system reaches the DFE state where the total population \( N^* \) is given by (3.2).

We will find the solution of these simplified equations (4.1)-(4.2) under a periodic impulsive vaccination showing that it exhibits a periodic steady regimen of period \( T = T(m, \sigma) = mt_v + \sigma \), with \( \sigma \in [0, t_v) \), \( m \in \mathbb{N} \cup \{0\} \). Furthermore, we will obtain the maximum values of the susceptible and vaccinated subpopulations within such a periodic regime. Proposition 1 establishes that the period \( T(m, \sigma) \) of such a solution must be always a multiple of \( t_v \). Then, Proposition 2 proves that such a period is always \( t_v \).

**Proposition 1.** (Proposition 2 (i) of [14]) For a general periodic solution of (4.1)-(4.2) with a time period \( T = T(m, \sigma) = mt_v + \sigma \) it is required that \( \sigma = 0 \).

**Proposition 2.** There is a unique general periodic solution of (4.1)-(4.2) with time period \( T = T(1, 0) = t_v \). This solution would be, from Proposition 2, that with the smallest time period.

**Proof.** Assuming that a generalized periodic solution of (4.1)-(4.2) with time period \( T(n, 0) = nt_v \) exists, we proceed to obtain the possible values for \( n \). In order to simplify the notation, we define the variables for the vaccinated and susceptible subpopulations within the interval between two consecutive impulses after a large enough time so that they have reached the periodic regime and the total population is constant, namely \( N(t) = N^* \). The susceptible and vaccinated subpopulations in such a situation can be denoted by:

\[
\forall \{i, r\} \in \mathbb{N}_0 \triangleq \mathbb{N} \cup \{0\} \ , \tau \in [0, t_v) \rightarrow \begin{align*}
S_i(\tau) &\triangleq \lim_{r \to \infty} S'(\tau + (i + r)t_v) \\
V_i(\tau) &\triangleq \lim_{r \to \infty} V'(\tau + (i + r)t_v)
\end{align*} \tag{4.4}
\]

The periodicity requires that:

\[
S_{i+n}(\tau) = S_i(\tau) \ , \ V_{i+n}(\tau) = V_i(\tau) , \quad \forall \tau \in [0, t_v) \tag{4.5}
\]

Also, the delayed vaccinated subpopulation term of (4.1) for \( \omega = kt_v + xt_v \), where \( k \in \mathbb{N}_0 \) and \( x \in [0, 1) \cap \mathbb{R} \), can be written as:

\[
V_i(\tau - \omega) = V_i(\tau - ((k + x)t_v)) = \begin{cases} 
V_{i-(k+1)}(\tau + (1 - x)t_v) & 0 \leq \tau < xt_v \\
V_{i-k}(\tau - xt_v) & xt_v \leq \tau < t_v
\end{cases} \tag{4.6}
\]

and with this equation the susceptible subpopulation dynamics in the periodic state can be written as:

\[
\dot{S}_i(\tau) = \begin{cases} 
b_1 - b_2 S_i(\tau) + \gamma_1 V_{i-(k+1)}(\tau + (1 - x)t_v)e^{-b_2\omega} + b_3(1 - V_c)N^* & 0 \leq \tau < xt_v \\
b_1 - b_2 S_i(\tau) + b_3(1 - V_c)N^* + \gamma_1 V_{i-k}(\tau - xt_v)e^{-b_2\omega} & xt_v \leq \tau < t_v
\end{cases} \tag{4.7}
\]
while the equation for the dynamics of the vaccinated subpopulation is:

\[
\dot{V}_i(t) = -\gamma_1 V_i(t) - b_2 V_i(t) + b_3 V_i(t) N^* \tag{4.8}
\]

By solving this equation between two consecutive impulses, one obtains:

\[
V_i(t) = (V_i(0^+) - V^*) e^{-(b_2 + \gamma_1) t} + V^* (1 - e^{-(b_2 + \gamma_1) t}) + V_i(0^+) e^{-(b_2 + \gamma_1) t} \tag{4.9}
\]

with \( V^* \) defined in (3.4). Then, using this result, we obtain the time-evolution for the periodic solution of the susceptible subpopulation from (4.7):

\[
S_i(t) = \begin{cases}
  e^{-b_2(t+\omega)} (V^*(e^{b_2 t}-1)\gamma_i/b_2) \\
  + (V_{i-k+1}(0^+)-V^*)e^{-(b_2+\gamma_1)(t+(1-x)t_v)}(1-e^{\gamma_1 t}) \\
  + (1-e^{-b_2 t})S^* + e^{-b_2 t}S_{i-1}(0^+) & 0 \leq t < x t_v \\
  e^{-b_2(t+\omega)} ((V_i-k(0^+)-V^*)e^{b_2 t_v} (1-e^{\gamma_1 (x t_v-\tau)})) \\
  + (V_{i-k+1}(0^+)-V^*)e^{-(b_2+\gamma_1) t_v (1-x)} (1-e^{-(b_2 + \gamma_1) t_v}) \\
  \left(1-e^{-b_2 \tau}\right) S^* + e^{-b_2 \tau} S_{i-1}(0^*) + e^{-(b_2 + \omega) t_v} (e^{b_2 t_m - 1}) \gamma_i/b_2 & x t_v \leq t < t_v
\end{cases} \tag{4.10}
\]

with \( S^* \) defined in (3.3). From (4.2) and the time evolutions (4.9),(4.10) at \( t = t_v \), we can describe the relation between the different values of the subpopulations after different impulses as:

\[
\begin{cases}
  S_i(0^+) = (1-\theta)S_{i-1}(t_v) \\
  S_i(t_v) = a S_i(0^+) + b V_{i-k}(0^+) + c V_{i-(k+1)}(0^+) + d \\
  V_i(t_v) = C_{v0} + C_{v1} V_i(0^+) \\
  V_i(0^+) = V_i-1(t_v) + \theta S_{i-1}(t_v)
\end{cases} \tag{4.11}
\]

being \( C_{v1} = e^{-(b_2 + \gamma_1) t_v}, C_{v0} = V^* (1-C_{v1}) \) and \( a = e^{-b_2 t_v}, b = C_{v1} e^{-b_2 (\omega - x t_v)} (e^{\gamma_1 t_v} - e^{\gamma_1 x t_v}) \), \( c = a C_{v1} e^{-b_2 (\omega - x t_v)} (e^{\gamma_1 x t_v} - 1) \) and \( d = (1-a) S^* - (b+c+(1-a)) e^{-b_2 \omega \gamma_1 / b_2} V^* \).

As we see in (4.5) the relations in (4.11) describe a n-cycle. Then, such equations can be presented in matrix form:
with the row switching matrix \( R = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \) which satisfies \( R \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \). We rearrange (4.15) and (4.14) using (4.13), so we get:

\[
\begin{pmatrix} V_1(0^+) \\ V_2(0^+) \\ \vdots \\ V_n(0^+) \end{pmatrix} = R \begin{pmatrix} C_{v0} \\ C_{v0} \\ \vdots \\ C_{v0} \end{pmatrix} + C_{v1} R \begin{pmatrix} V_1(0^+) \\ V_2(0^+) \\ \vdots \\ V_n(0^+) \end{pmatrix} + \theta R \begin{pmatrix} S_1(t_v) \\ S_2(t_v) \\ \vdots \\ S_n(t_v) \end{pmatrix}
\]

(4.16)

\[
= R \begin{pmatrix} C_{v0} \\ C_{v0} \\ \vdots \\ C_{v0} \end{pmatrix} + C_{v1} R \begin{pmatrix} V_1(0^+) \\ V_2(0^+) \\ \vdots \\ V_n(0^+) \end{pmatrix} + \theta R \begin{pmatrix} S_1(0^+) \\ S_2(0^+) \\ \vdots \\ S_n(0^+) \end{pmatrix} + R^k (bI + cR) \begin{pmatrix} V_1(0^+) \\ V_2(0^+) \\ \vdots \\ V_n(0^+) \end{pmatrix}
\]

Then we get the values of the susceptible subpopulation after the impulsive time instants in relation to the values of the vaccinated subpopulation at such time instants:

\[
\begin{pmatrix} S_1(0^+) \\ S_2(0^+) \\ \vdots \\ S_n(0^+) \end{pmatrix} = \frac{1}{\theta a} \left( R^{-1} - C_{v1} I - \theta R^k (bI + cR) \right) \begin{pmatrix} V_1(0^+) \\ V_2(0^+) \\ \vdots \\ V_n(0^+) \end{pmatrix} - (C_{v0} + \theta d) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}
\]

(4.17)

Note that if \( \theta = 1 \) then, from (4.12), we know that \( S_i(0^+) = 0 \forall i \in \mathbb{N} \), and it follows from (4.17) that:

\[
(C_{v0} + d) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = (R^{-1} - C_{v1} I - R^k (bI + cR)) \begin{pmatrix} V_1(0^+) \\ V_2(0^+) \\ \vdots \\ V_n(0^+) \end{pmatrix}
\]

(4.18)

We define a matrix \( M_0 \) as:

\[
M_0 = (C_{v0} + d)^{-1} \left( R^{-1} - C_{v1} I - R^k (bI - cR) \right)
\]

(4.19)

Note that the sum of all the elements which compose any row in the matrices \( R^k, \forall k \in \mathbb{Z} \), is equal to 1. Such a fact implies that:

\[
M_0 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} m_0 \\ m_0 \\ \vdots \\ m_0 \end{pmatrix}
\]

(4.20)
being \( m_0 = \frac{1-C_{v0} - b-c}{C_{v0} + d} \). As \( M_0 \) is row-equivalent to \( I \), we know from [18] that:

\[
\exists M_0^{-1} \rightarrow M_0^{-1}M_0 \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}
\]

(4.21)

Then, from (4.18) we get:

\[
\begin{pmatrix} V_1(0^+) \\ V_2(0^+) \\ \vdots \\ V_n(0^+) \end{pmatrix} = M_0^{-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \frac{1}{m_0} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}
\]

(4.22)

proving that \( \{ V_j(0^+) = V_i(0^+) \}, S_j(0^+) = S_i(0^+) \}, \forall i, j \in \mathbb{N}, \) for \( \theta = 1 \).

If \( \theta \neq 1 \), we get \( S_i(t_v) \) and \( V_i(t_v) \) from (4.12) and (4.15), respectively, and by applying them on (4.14) it follows that:

\[
\begin{pmatrix} V_1(0^+) \\ V_2(0^+) \\ \vdots \\ V_n(0^+) \end{pmatrix} = R \begin{pmatrix} C_{v0} \\ C_{v0} \\ \vdots \\ C_{v0} \end{pmatrix} + C_{v1}R \begin{pmatrix} V_1(0^+) \\ V_2(0^+) \\ \vdots \\ V_n(0^+) \end{pmatrix} + \frac{\theta}{1-\theta} \begin{pmatrix} S_1(0^+) \\ S_2(0^+) \\ \vdots \\ S_n(0^+) \end{pmatrix}
\]

(4.23)

Then, by applying (4.17) in (4.23) we get:

\[
\begin{pmatrix} V_1(0^+) \\ V_2(0^+) \\ \vdots \\ V_n(0^+) \end{pmatrix} = \left( C_{v0} - \frac{C_{v0} + \theta d}{a(1-\theta)} \right) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}
\]

(4.24)

\[
+ \left( C_{v1}R + \frac{R^{-1} - C_{v1}I - \frac{\theta R^{k+1}}{a(1-\theta)}(bR^{-1} + cI)}{a(1-\theta)} \right) \begin{pmatrix} V_1(0^+) \\ V_2(0^+) \\ \vdots \\ V_n(0^+) \end{pmatrix}
\]

so the values of the vaccinated subpopulation after the impulsive time instants are defined as:

\[
\begin{pmatrix} V_1(0^+) \\ V_2(0^+) \\ \vdots \\ V_n(0^+) \end{pmatrix} = \left[ C_{v0} - \frac{C_{v0} + \theta d}{a(1-\theta)} \right] \left[ I - \left( C_{v1}R + \frac{R^{-1} - C_{v1}I - \frac{\theta R^{k+1}}{a(1-\theta)}(M + cI)}{a(1-\theta)} \right) \right]^{-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}
\]

(4.25)

We define a matrix \( M \) as:

\[
M = \left[ C_{v0} - \frac{C_{v0} + \theta d}{a(1-\theta)} \right]^{-1} \left[ I - \left( C_{v1}R + \frac{R^{-1} - C_{v1}I - \frac{\theta R^{k+1}}{a(1-\theta)}(M + cI)}{a(1-\theta)} \right) \right]
\]

(4.26)
Note that the sum of all the elements which compose any row in the matrices $R^k, \forall k \in \mathbb{Z}$, is equal to 1. Such a fact implies that:

$$M \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = m \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

(4.27)

being $m = \frac{1 - C_v - 1 - C_v - \theta(b+c)}{a(1-\theta)} / \left( C_{v0} - \frac{C_v + \theta d}{a(1-\theta)} \right)$. Since, as in the previous demonstration, $M$ is row equivalent to $I$, then we know from [18] that is invertible so that:

$$\exists M^{-1} \rightarrow M^{-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

(4.28)

so the values of the vaccinated subpopulation after the regular impulsive vaccination instants are:

$$\begin{pmatrix} V_1(0^+) \\ V_2(0^+) \\ \vdots \\ V_n(0^+) \end{pmatrix} = M^{-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \frac{1}{m} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

(4.29)

and, from (4.17), the susceptible subpopulation after such impulsive instants are:

$$\begin{pmatrix} S_1(0^+) \\ S_2(0^+) \\ \vdots \\ S_n(0^+) \end{pmatrix} = \frac{1}{a\theta} \left[ \frac{1 - C_v - \theta (b+c)}{m} - (C_{v0} + \theta d) \right] \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

(4.30)

In summary, it is established that, either if $\theta = 1$ or $\theta \in (0,1)$ the values of the susceptible and vaccinated subpopulations just after any impulsive vaccination instant at the periodic regime will be the same, i.e : \{ $S_i(0^+) = S_j(0^+), V_i(0^+) = V_j(0^+) \}$, $\forall i, j \in \mathbb{N}$ As the time evolutions of the susceptible and the vaccinated subpopulations within the time period between any two consecutive impulsive vaccination instants depend only in such initial values (see (4.10) and (4.9) then \{ \begin{align*} S_i(\tau) &= S_j(\tau), \\ V_i(\tau) &= V_j(\tau) \end{align*} \} $\forall \tau \in [0, t_v)$, $\forall j, i \in \mathbb{N}$ The matrices $M$ and $M_0$ are invertible so this solution is unique for any $T(n,0) = nt_v$. We see in (4.3) that, if $b_2 > b_3$, after a sufficiently large time $t_0$ the total population reaches a constant value $N(t) = N^* \forall t > t_0$, so the recovered subpopulation $R_i(\tau) = N^* - (S_i(\tau) + V_i(\tau))$ must also present the same periodicity that the susceptible and vaccinated subpopulations. Therefore, from equations (4.29) and (4.30), the solution of the generalized periodic solution of the subsystem of subpopulations $S'$, $V'$ and $R'$ is periodic and present the smallest periodic solution with time period $T(1,0)$. \qed
5 Other models

While the disease free periodic state is valid for the model proposed, it is also valid for many other possible models for which the SVEIR is just a particular case. A more generalist model of that dealt with in sections 1-4 is presented here where a series of functions replace the infective influence of the dynamic of those appearing in the first model by using distributions on a certain period of time associated with a periodicity of the disease evolution. Furthermore, the number of latency stages of the disease is not necessarily prefixed a priori. It can be noticed an indeterminate number of non-infectious stages and an arbitrary delay distributed process of transmission and transition between the subpopulations:

\[
\begin{align*}
\dot{S}(t) &= b_1 - b_2 S(t) + \gamma_1 V(t - \omega) e^{-b_2 \omega} + b_3 (1 - V_c) N(t) + f_1(t) \\
\dot{V}(t) &= b_3 V_c N(t) - (\gamma_1 + b_2) V(t) + f_2(t) \\
\dot{E}_j(t) &= f_{j+2}(t) - b_2 E_j(t) \\
\dot{I}(t) &= f_{m+3}(t) - b_2 I(t) \\
\dot{R}(t) &= \gamma_1 V(t) - b_2 R(t) - \gamma_1 V(t - \omega) e^{-b_2 \omega} + f_{m+4}(t)
\end{align*}
\]

\[\forall j = 1, 2, ..., m\]

\[
\begin{cases}
S(t^+) = (1 - \theta)S(t) \\
V(t^+) = V(t) + \theta S(t) & \text{if } t = nt_v \ (n = 1, 2, 3, ...) \\
R(t^+) = R(t) & \theta \in [0, 1] \\
E_j(t^+) = E_j(t) \\
I(t^+) = I(t)
\end{cases}
\] (5.1)

with \( f_i(t) = p_i(t) I(t) + \int_0^t h_i(t') g_i(t - t') dt' \), being the \( h_i(t) \) generic delay distribution functions, and the functions:

\[
\begin{align*}
p_i(t) &= p_i (S(t), V(t), E_1(t), ..., E_m(t), I(t), R(t)) \\
g_i(t) &= g_i (S(t), V(t), E_1(t), ..., E_m(t), I(t), R(t)) \ , \ \forall i \in [1, m + 4]
\end{align*}
\]

are defined so that the terms \( f_i(t) = 0 \) if the infected subpopulations are zero for a time interval \( E_1(t') = E_2(t') = ... = E_m(t') = I(t') = 0 \ \forall t' \in [t - \tau, t] \).

**Example 5.1** An example of this type of model would be a simple \( SV E_m IR \) model with \( m = 1 \) and no additional delays in the infected subpopulations while the immunity delay remain, in which the distribution function \( h_i(t) = \)
being $\delta(t)$ the dirac delta function. Thus, the equations of the continuous part of the model are

\[
\begin{align*}
\dot{S}(t) &= b_1 + b_3 (1-V_c) N(t) - \beta S(t) I(t) + e^{-b_2 \omega} \gamma_1 V(t - \omega) + \gamma I(t - \omega) e^{-b_2 \omega} - b_2 S(t) \\
\dot{V}(t) &= b_3 V_c N(t) - \beta' V(t) I(t) - (\gamma_1 + b_2) V(t) \\
\dot{E}(t) &= (\beta S(t) + \beta' V(t)) I(t) - (b_2 + \kappa) E(t) \\
\dot{I}(t) &= \kappa E(t) - (\gamma + \alpha + b_2) I(t) \\
\dot{R}(t) &= \gamma_1 \left( V(t) - V(t - \omega) e^{-b_2 \omega} \right) + \gamma \left( I(t) - I(t - \omega) e^{-b_2 \omega} \right) - b_2 R(t)
\end{align*}
\]  

(5.3)

with the impulsive part being the same as in (2.6)

**Example 5.2** Another example would be a model with the same infection rate as in section 2, but with a serie of $m$ stages of the disease reaching to the infectious subpopulation instead of just only a exposed subpopulation, the functions would be defined then as:

\[
\begin{align*}
\left\{ \begin{array}{l}
\begin{array}{l}
f_1(t) = -\frac{\beta S(t) I(t)}{1 + \eta S(t)} + \int_0^t \frac{\gamma e^{-b_2 t'} \delta(t' - \omega) I(t - t') dt'}{1 + \eta S(t)} \\
f_2(t) = -\frac{\beta' V(t) I(t)}{1 + \eta V(t)} \\
\end{array} \\
\begin{array}{l}
f_{j+2}(t) = \int_0^t \delta(t' - \tau_{j-1}) - \delta(t - \tau_j) e^{-b_2 t'} \beta \left[ \frac{\eta S(t-t') I(t-t')}{1 + \eta S(t-t')} + \frac{\eta V(t-t') I(t-t')}{1 + \eta V(t-t')} \right] dt' \\
f_{m+3}(t) = \int_0^t \delta(t' - \tau_m) e^{-b_2 t'} \beta \left[ \frac{\eta S(t-t') I(t-t')}{1 + \eta S(t-t')} + \frac{\eta V(t-t') I(t-t')}{1 + \eta V(t-t')} \right] dt' - (\alpha + \gamma) I(t) \\
f_{m+4}(t) = \int_0^t \gamma e^{-b_2 t'} (\delta(t) - \delta(t' - \omega)) I(t - t') dt'
\end{array}
\end{array} \right.
\]  

(5.4)
for \(j=1, 2, \ldots m\), with \(0 = \tau_0 < \tau_1 < \tau_2 < \tau_3 < \ldots < \tau_{m-1} < \tau_m\) and \(\delta(t)\) the delta dirac function. The equations of the model would be

\[
\begin{align*}
\dot{S}(t) &= b_1 - b_2 S(t) - \frac{\beta S(t)I(t)}{1+\eta S(t)} + e^{-b_2 \omega} (\gamma I(t - \omega) + \gamma_1 V(t - \omega)) + b_3 (1 - V_c) N(t) \\
\dot{V}(t) &= -\beta \frac{V(t)I(t)}{1+\eta V(t)} \gamma_1 V(t) - b_2 V(t) + b_3 V_c N(t) \\
\dot{E}_1(t) &= \beta \left[ \frac{S(t)I(t)}{1+\eta S(t)} + \rho \frac{V(t)I(t)}{1+\eta V(t)} - e^{-b_2 \tau_1} \left( \frac{S(t-\tau_1)I(t-\tau_1)}{1+\eta S(t-\tau_1)} + \rho \frac{V(t-\tau_1)I(t-\tau_1)}{1+\eta V(t-\tau_1)} \right) \right] \\
& \quad - b_2 E_1(t) \\
& \quad \ldots \\
\dot{E}_j(t) &= \beta \left[ e^{-b_2 \tau_{j-1}} \left( \frac{S(t-\tau_{j-1})I(t-\tau_{j-1})}{1+\eta S(t-\tau_{j-1})} + \rho \frac{V(t-\tau_{j-1})I(t-\tau_{j-1})}{1+\eta V(t-\tau_{j-1})} \right) \\
& \quad - e^{-b_2 \tau_j} \left( \frac{S(t-\tau_j)I(t-\tau_j)}{1+\eta S(t-\tau_j)} + \rho \frac{V(t-\tau_j)I(t-\tau_j)}{1+\eta V(t-\tau_j)} \right) \right] - b_2 E_j(t) \\
\dot{I}(t) &= \beta e^{-b_2 \tau_m} \left( \frac{S(t-\tau_m)I(t-\tau_m)}{1+\eta S(t-\tau_m)} + \rho \frac{V(t-\tau_m)I(t-\tau_m)}{1+\eta V(t-\tau_m)} \right) - (b_2 + \alpha + \gamma) I(t) \\
\dot{R}(t) &= \gamma_1 V(t) + \gamma I(t) - b_2 R(t) - (\gamma I(t - \omega) + \gamma_1 V(t - \omega)) e^{-b_2 \omega} \tag{5.5}
\end{align*}
\]

for \(j=1, 2, \ldots m\) with the impulsive part being similar to that of (2.6). From (5.1) it is seen that, although the SVEIRmIR dynamics in general is quite different, for \(f_i = 0\) all the subpopulations tend to the same 3-dimensional hyperplane containing the disease free solution as in (4.1) and (4.2), and the solution for this type situation can all be described with similar terms.

**Acknowledgments**

The authors thank the Spanish Ministry of Education support through Grant BES-2010-035160 of the project DPI2009-07197, Grant DPI2012- 30651, and the Basque Government for its support trough Grants IT378-10 and SAIOTEK S-PE12UN015 and S-PE13UN039. They are also grateful to the UPV/EHU for its support through Grant UFI 2011/07.

**References**


Received: December 1, 2013