Laplace Type Problem for Lattice with Cell Composed by Two Quadrilaterals and One Triangle

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Abstract

In some previous papers [1], [2], [3], [4] the authors studies some Laplace type problems for some lattice with different fundamental cell. We want to compute the probability that a segment of random position and of constant lenght intersects a side of a lattice with cell rapresented in fig.1.

1 Main Results

Let $\mathfrak{R}(a)$ the lattice with the fundamental cell C_0 rapresented in fig.1

fig.1

We have

$$C_0 = C_{01} \cup C_{02} \cup C_{03}$$

and

$$\widehat{BAD} = \widehat{FAD} = \frac{3\pi}{10}, \quad \widehat{FEG} = \widehat{FGE} = \frac{2\pi}{5}, \quad \widehat{EFG} = \frac{\pi}{5} \quad (1)$$

$$|AD| = a \sin \frac{\pi}{5} \left(1 + 2 \cos \frac{\pi}{5}\right), \quad |CD| = |DE| = \frac{a}{2}, \quad |EG| = \frac{a}{2 \cos \frac{\pi}{5}} \quad (2)$$

$$\text{area} C_0 = \frac{a^2}{2} \sin \frac{\pi}{5} \cdot \left(2 \cos \frac{\pi}{5} + 1\right)^2 \quad (3)$$

We want to compute the probability that a segment s of random position and constant length l , with $l < \frac{a}{4}$, intersect a side of the lattice $\mathfrak{R}(a)$, i.e. the probability P_{int} that s intersect a side of the fundamental cell C_0 .

The position of the segment s is determined by middle point and by the angle φ that is formed with the line CG .

In order to compute P_{int} we consider the limiting positions of the segment s , for a specified value of φ , in the cell C_{0i} , ($i = 1, 2, 3$). We have the fig. 2.

fig.2

and the relations

$$areaC_{01}(\varphi) = areaC_{01} - \sum_{j=1}^6 areaa_j(\varphi) \quad (4)$$

$$area\hat{C}_{02}(\varphi) = areaC_{02} - \sum_{j=1}^6 areab_j(\varphi) \quad (5)$$

$$area\hat{C}_{03}(\varphi) = areaC_{03} - \sum_{j=1}^5 areac_j(\varphi) \quad (6)$$

We consider the fig.3

fig.3

we have:

$$|DD_1| = l \cos \varphi, \quad |DD_2| = l \sin \varphi \quad (7)$$

then

$$areaa_5(\varphi) = \frac{l^2}{4} \sin 2\varphi. \quad (8)$$

By fig.4

fig.4

follows

$$h_4 = \frac{l}{2} \sin \varphi.$$

Then, with first relation (7), we have

$$|CD_1| = \frac{a}{2} - |DD_1| = \frac{a}{2} - l \cos \varphi.$$

Then

$$areaa_4(\varphi) = \frac{al}{4} \sin \varphi - \frac{l^2}{4} \sin 2\varphi. \quad (9)$$

The relations (8) and (9) give us

$$areaa_4(\varphi) + areaa_5(\varphi) = \frac{al}{4} \sin \varphi \quad (10)$$

Considering the fig.5

fig.5

we can write

$$\widehat{BBC'} = \frac{3\pi}{5} - \varphi, \quad h_3 = \frac{l}{2} \sin \left(\frac{2\pi}{5} + \varphi \right),$$

then

$$areaa_3(\varphi) = \frac{al}{2} \sin \left(\frac{2\pi}{5} + \varphi \right). \quad (11)$$

Now we consider the fig.6

fig.6

we have

$$\begin{aligned} \widehat{AA_2A_1} &= \widehat{DD_2D_1} = \pi - \widehat{D_1D_2D} = \\ \pi - \left(\frac{\pi}{2} - \varphi \right) &= \frac{\pi}{2} + \varphi, \quad \widehat{A_1AA_2} = \frac{\pi}{5} - \varphi, \\ \widehat{A_1AA_2} &= \frac{3\pi}{10}. \end{aligned} \quad (12)$$

With these angles, the triangle AA_1A_2 give us

$$|AA_1| = \frac{l \cos \varphi}{\cos \frac{\pi}{5}}, \quad |AA_2| = \frac{l \sin \left(\frac{\pi}{5} - \varphi \right)}{\cos \frac{\pi}{5}} \quad (13)$$

then

$$areaa_1(\varphi) = \frac{l^2 \cos \varphi \sin \left(\frac{\pi}{5} - \varphi \right)}{2 \cos \frac{\pi}{5}} \quad (14)$$

Therefore by (12) we have that

$$\varphi \leq \frac{\pi}{5} \quad (15)$$

By fig. 7

fig.7

and combining (12) and (13) we obtain that

$$\widehat{ABB'} = \widehat{AA_1A_2} = \frac{\pi}{5} - \varphi,$$

$$h_2 = \frac{l}{2} \sin \left(\frac{\pi}{5} - \varphi \right),$$

$$|A_1B| = a - |AA_1| = a - \frac{l \cos \varphi}{\cos \frac{\pi}{5}}.$$

consequently

$$areaa_2(\varphi) = \frac{al}{2} \sin \left(\frac{\pi}{5} - \varphi \right) - \frac{l^2 \cos \varphi \cos \left(\frac{\pi}{5} - \varphi \right)}{2 \cos \frac{\pi}{5}} \quad (16)$$

The relations (14) and (16) give us

$$areaa_1(\varphi) + areaa_2(\varphi) = \frac{al}{2} \sin \left(\frac{\pi}{5} - \varphi \right). \quad (17)$$

The fig. 8

fig.8

give us

$$\widehat{A'D'D_2} = \widehat{DD_2D_1} = \frac{\pi}{2} - \varphi, \quad h_6 = \frac{l}{2} \cos \varphi$$

Then, considering the formulas (2), (7) and (13) follows

$$|A_2D_2| = |AD| - |AA_2| - |DD_2| =$$

$$a \sin \frac{\pi}{5} \left(1 + \cos \frac{\pi}{5}\right) - l \sin \varphi - \frac{l \sin \left(\frac{\pi}{5} - \varphi\right)}{\cos \frac{\pi}{5}}.$$

Then

$$areaa_6(\varphi) = \frac{al}{2} \sin \frac{\pi}{5} \left(1 + \cos \frac{\pi}{5}\right) \cos \varphi - \frac{l^2}{4} \sin 2\varphi - \frac{l^2 \cos \varphi \sin \left(\frac{\pi}{5} - \varphi\right)}{2 \cos \frac{\pi}{5}} \quad (18)$$

Replacing in (4) the relations (10),(11), (17) and (18) we obtain

$$area\widehat{C}_{01}(\varphi) = areaC_{01} - A_1(\varphi), \quad (19)$$

where

$$A_1(\varphi) = \frac{al}{2} \left[\sin \frac{\pi}{5} \left(2 + 3 \cos \frac{\pi}{5}\right) \cos \varphi + \right.$$

$$\left. \left(2 \cos^2 \frac{\pi}{5} - \cos \frac{\pi}{5} - \frac{1}{2}\right) \sin \varphi \right] - \frac{l^2}{4} \tan \frac{\pi}{5} (1 + \cos 2\varphi). \quad (20)$$

Now we consider the fig.9

fig.9

we have

$$\widehat{E_1EE_2} = \frac{3\pi}{5}, \quad \widehat{EE_2E_1} = \frac{2\pi}{5} - \varphi \quad (21)$$

With these angles the triangle EE_1E_2 give us

$$|EE_1| = \frac{l \sin \left(\frac{2\pi}{5} - \varphi \right)}{\sin \frac{2\pi}{5}}, \quad |EE_2| = \frac{l \sin \varphi}{\sin \frac{2\pi}{5}} \quad (22)$$

then

$$areab_4(\varphi) = \frac{l^2 \sin \varphi \sin \left(\frac{2\pi}{5} - \varphi \right)}{2 \sin \frac{2\pi}{5}} \quad (23)$$

By fig.10

fig.10

and from first formula (22) follows

$$h_3 = \frac{l}{2} \sin \varphi, \quad |DE_1| = \frac{a}{2} - |EE_1| = \frac{a}{2} - \frac{l \sin \left(\frac{2\pi}{5} - \varphi \right)}{\sin \frac{2\pi}{5}},$$

then

$$areab_3(\varphi) = \frac{al}{4} \sin \varphi - \frac{l^2 \sin \varphi \left(\frac{2\pi}{5} - \varphi \right)}{2 \sin \frac{2\pi}{5}} \quad (24)$$

The relations (23) and (24) give us

$$areab_3(\varphi) + areab_4(\varphi) = \frac{al}{4} \sin \varphi \quad (25)$$

The fig.11

fig.11

give us

$$\widehat{A_3AA_4} = \frac{3\pi}{10}, \quad \widehat{AA_3A_4} = \frac{\pi}{2} - \varphi, \quad \widehat{AA_4A_3} = \frac{\pi}{5} + \varphi, \quad (26)$$

then

$$|AA_3| = \frac{l \sin\left(\frac{\pi}{5} + \varphi\right)}{\cos \frac{\pi}{5}}, \quad |AA_4| = \frac{l \cos \varphi}{\cos \frac{\pi}{5}} \quad (27)$$

consequently

$$areab_1(\varphi) = \frac{l^2 \cos \varphi \sin\left(\frac{\pi}{5} + \varphi\right)}{2 \cos \frac{\pi}{5}}. \quad (28)$$

Now we consider the fig.12

fig.12

We have

$$\widehat{A_3DD''} = \frac{\pi}{2} - \varphi, \quad h_2 = \frac{l}{2} \cos \varphi.$$

Then, by combining (2) and (27) we see that

$$|A_3D| = |AD| - |AA_3| = a \sin \frac{\pi}{5} \left(1 + 2 \cos \frac{\pi}{5}\right) - \frac{l \sin\left(\frac{\pi}{5} + \varphi\right)}{\cos \frac{\pi}{5}}.$$

Then

$$areab_2(\varphi) = \frac{al}{2} \sin \frac{\pi}{5} \left(1 + 2 \cos \frac{\pi}{5}\right) \cos \varphi - \frac{l^2 \cos \varphi \sin\left(\frac{\pi}{5} + \varphi\right)}{2 \cos \frac{\pi}{5}}$$

This relation and the (28) give us

$$areab_1(\varphi) + areab_2(\varphi) = \frac{al}{2} \sin \frac{\pi}{5} \left(1 + 2 \cos \frac{\pi}{5}\right) \cos \varphi. \quad (29)$$

The fig.13

fig.13

and the formulas (21) and (22) give us

$$\widehat{F''E''E_2} = \widehat{EE_2E_1} = \frac{2\pi}{5} - \varphi, \quad h_5 = \frac{l}{2} \sin \left(\frac{2\pi}{5} - \varphi \right)$$

and

$$|FE_2| = a - |EE_2| = a - \frac{l \sin \varphi}{\sin \frac{2\pi}{5}}.$$

Therefore

$$areab_5(\varphi) = \frac{al}{2} \sin \left(\frac{2\pi}{5} - \varphi \right) - \frac{l^2 \sin \varphi \left(\frac{2\pi}{5} - \varphi \right)}{2 \sin \frac{2\pi}{5}} \quad (30)$$

At last we consider the fig.14

fig.14

Combining (26) and (27) we see that

$$\widehat{F''A''A_4} = \widehat{AA_4A_3} = \frac{\pi}{5} + \varphi, \quad h_6 = \frac{l}{2} \sin \left(\frac{\pi}{5} + \varphi \right),$$

$$|A_4F| = a - |AA_4| = a - \frac{l \cos \varphi}{\cos \frac{\pi}{5}},$$

then

$$area b_6(\varphi) = \frac{al}{2} \sin \left(\frac{\pi}{5} + \varphi \right) - \frac{l^2 \cos \varphi \sin \left(\frac{\pi}{5} + \varphi \right)}{2 \cos \frac{\pi}{5}}. \quad (31)$$

Replacing in the (5) the relations (25),(29),(30) and (31) we get

$$area C_{02}(\varphi) = area C_{02} - A_2(\varphi) \quad (32)$$

where

$$\begin{aligned} A_2(\varphi) = \frac{al}{4} \left[2 \sin \frac{\pi}{5} \left(1 + 6 \cos \frac{\pi}{5} \right) \cos \varphi + \sin \varphi \right] - \frac{l^2}{4 \sin \frac{2\pi}{5}} \\ \left[\left(\cos \frac{2\pi}{5} + \sin \frac{2\pi}{5} \right) \cos 2\varphi + \left(\cos \frac{2\pi}{5} + \right. \right. \\ \left. \left. \sin \frac{2\pi}{5} + 1 \right) \sin 2\varphi + \sin \frac{2\pi}{5} - \cos \frac{2\pi}{5} \right] \end{aligned} \quad (33)$$

Now we compute $area \widehat{C}_{03}(\varphi)$.

By the fig.15

fig.15

we have

$$\widehat{G_1GG_2} = \frac{2\pi}{5}, \quad \widehat{GG_1G_2} = \frac{3\pi}{5} - \varphi \quad (34)$$

With these angles the triangle GG_1G_2 give us

$$|GG_1| = \frac{l \sin \varphi}{\sin \frac{2\pi}{5}}, \quad |GG_2| = \frac{l \sin \left(\frac{2\pi}{5} + \varphi \right)}{\sin \frac{2\pi}{5}} \quad (35)$$

then

$$areac_4(\varphi) = \frac{l^2 \sin \varphi \sin \left(\frac{2\pi}{5} + \varphi \right)}{2 \sin \frac{2\pi}{5}} \quad (36)$$

Considering the fig.16

fig.16

and second formula (35) we obtain

$$h_3 = \frac{l}{2} \sin \varphi, \quad |EG_2| = |EG| - |GG_2| = \frac{a}{2 \cos \frac{\pi}{5}} - \frac{l \sin \left(\frac{2\pi}{5} + \varphi \right)}{\sin \frac{2\pi}{5}},$$

then

$$areac_3(\varphi) = \frac{al \sin \varphi}{4 \cos \frac{\pi}{5}} - \frac{l^2 \sin \varphi \sin \left(\frac{2\pi}{5} + \varphi \right)}{2 \sin \frac{2\pi}{5}}$$

This relation with (36) give us

$$areac_3(\varphi) + areac_4(\varphi) = \frac{al \sin \varphi}{4 \cos \frac{\pi}{5}} \quad (37)$$

By the fig.17

fig.17

and by second formula (34) we see that

$$\widehat{F_1 F F_2} = \frac{\pi}{5}, \quad \widehat{F F_1 F_2} = \frac{2\pi}{5} + \varphi, \quad \widehat{F F_2 F_1} = \frac{2\pi}{5} - \varphi.$$

With these angles the triangle FF_1F_2 give us

$$|FF_1| = \frac{l \sin\left(\frac{2\pi}{5} - \varphi\right)}{\sin \frac{\pi}{5}}, \quad |FF_2| = \frac{l \sin\left(\frac{2\pi}{5} + \varphi\right)}{\sin \frac{\pi}{5}} \quad (38)$$

then

$$areac_1(\varphi) = \frac{l^2}{4 \sin \frac{\pi}{5}} \left(\cos 2\varphi + \cos \frac{\pi}{5} \right). \quad (39)$$

The fig.18

fig.18

and the second relation (38) give us

$$h_2 = \frac{l}{2} \sin\left(\frac{2\pi}{5} - \varphi\right), \quad |EF_2| = a - |FF_2| = a - \frac{l \sin\left(\frac{2\pi}{5} + \varphi\right)}{\sin \frac{\pi}{5}},$$

then

$$areac_2(\varphi) = \frac{al}{2} \sin\left(\frac{2\pi}{5} - \varphi\right) - \frac{l^2}{4 \sin \frac{\pi}{5}} \left(\cos 2\varphi + \cos \frac{\pi}{5} \right). \quad (40)$$

Combining (39) and (40) we have

$$areac_1(\varphi) + areac_2(\varphi) = \frac{al}{2} \sin\left(\frac{2\pi}{5} - \varphi\right). \quad (41)$$

In the end we consider the fig.19

fig.19

With the second relation (34) we have

$$\widehat{F'G'G_1} = \widehat{GG_1G_2} = \frac{3\pi}{5} - \varphi, \quad h_5 = \frac{l}{2} \sin \left(\frac{2\pi}{5} + \varphi \right)$$

Then, considering the formules (35) and (38), we can write

$$|F_1G_1| = a - |FF_1| - |GG_1| = a - \frac{l \sin \left(\frac{2\pi}{5} - \varphi \right)}{\sin \frac{\pi}{5}} - \frac{l \sin \varphi}{\sin \frac{2\pi}{5}}$$

Consequently

$$\begin{aligned} \text{area}_{C_5}(\varphi) &= \frac{al}{2} \sin \left(\frac{2\pi}{5} + \varphi \right) - \frac{l^2}{4 \sin \frac{\pi}{5}} \left(\cos 2\varphi + \cos \frac{\pi}{5} \right) - \\ &\quad \frac{l^2}{4 \sin \frac{2\pi}{5}} \left(\sin \frac{2\pi}{5} \sin 2\varphi - \cos \frac{2\pi}{5} \cos 2\varphi + \cos \frac{2\pi}{5} \right). \end{aligned} \quad (42)$$

Replacing in the (6) the relations (37), (41) and (42) we get

$$\text{area} \widehat{C}_{03}(\varphi) = \text{area} C_{03} - A_3(\varphi), \quad (43)$$

where

$$\begin{aligned} A_3(\varphi) &= al \left(\sin \frac{2\pi}{5} \cdot \cos \varphi + \frac{\sin \varphi}{4 \cos \frac{\pi}{5}} \right) - \frac{l^2}{4 \sin \frac{2\pi}{5}} \\ &\quad \left[\left(2 \cos \frac{\pi}{5} - \cos \frac{2\pi}{5} \right) \cos 2\varphi + \sin \frac{2\pi}{5} \sin 2\varphi + 2 \cos \frac{2\pi}{5} + 1 \right]. \end{aligned} \quad (44)$$

Denoting with $M_i, (i = 1, 2, 3)$ the set of all segments s which their center in cell C_{01} . Denote likewise by N_i the set of segments s completely contained in C_{01} we have [2]:

$$P_{int} = 1 - \frac{\sum_{i=1}^3 \mu(N_i)}{\sum_{i=1}^3 \mu(M_i)}, \quad (45)$$

where μ is the Lebesgue measure in the euclidean plane.

To compute the above measure $\mu(M_i)$ and $\mu(N_i)$ we use the Poincaré[5] kinematic measure of]:

$$dK = dx \wedge dy \wedge d\varphi$$

Considering the formula (15) we can write

$$\begin{aligned} \mu(M_i) &= \int_0^{\pi/5} dy \int_{\{(x,y) \in C_{01}\}} dx dy = \\ \int_0^{\pi/5} (area C_{01}) dy &= \frac{\pi}{5} area C_{0i}, \quad (i = 1, 2, 3), \end{aligned}$$

then

$$\sum_{i=1}^3 \mu(M_i) = \frac{\pi}{5} area C_0. \quad (46)$$

In the same way, with the relations (19),(32) and (43) we have

$$\begin{aligned} \mu(N_i) &= \int_0^{\pi/5} d\varphi \int \int_{\{(x,y) \in \widehat{C_{0i}(\varphi)}\}} = \\ \int_0^{\pi/5} [area \widehat{C_{0i}(\varphi)}] dy &= \frac{\pi}{5} area C_{0i} - \int_0^{\pi/5} A_i(\varphi) dy. \end{aligned}$$

Then

$$\sum_{i=1}^3 \mu(N_i) = \frac{\pi}{5} area C_0 - \int_0^{\pi/5} \left[\sum_{i=1}^3 A_i(\varphi) \right] dy \quad (47)$$

Combining (20),(33) and (44) we have

$$\begin{aligned} \sum_{i=1}^3 A_i(\varphi) &= \frac{al}{2} \left[\sin \frac{\pi}{5} \left(13 \cos \frac{\pi}{5} + 3 \right) \cos \varphi + \frac{1}{2} \left(4 \cos^2 \frac{\pi}{5} - 2 \cos \frac{\pi}{5} + 3 + \frac{1}{\cos \frac{\pi}{5}} \right) \sin \varphi \right] - \\ \frac{l^2}{4 \sin \frac{2\pi}{5}} &\left[\left(1 + 2 \cos \frac{\pi}{5} - \cos \frac{2\pi}{5} \right) \cos 2\varphi + 2 \cos \frac{\pi}{5} \left(\cos \frac{\pi}{5} + 2 \sin \frac{\pi}{5} \right) \sin 2\varphi + \sin \frac{2\pi}{5} + 2 \right]. \end{aligned}$$

By this formula we obtain that

$$\begin{aligned}
\int_0^{\pi/5} \left[\sum_{i=1}^3 A_i(\varphi) \right] dy = \frac{al}{2} \left(1 - \cos \frac{\pi}{5} \right) \left(15 \cos^2 \frac{\pi}{5} + \right. \\
\left. 15 \cos \frac{\pi}{5} + \frac{9}{2} + \frac{1}{2 \cos \frac{\pi}{5}} \right) - \frac{l^2}{4 \sin \frac{2\pi}{5}} \left[\frac{1}{2} \sin \frac{2\pi}{5} \left(\sin \frac{2\pi}{5} - \right. \right. \\
\left. \left. \cos \frac{2\pi}{5} + 2 \cos \frac{\pi}{5} + 3 \right) + \frac{\pi}{5} \left(\sin \frac{2\pi}{5} + 2 \right) \right] \quad (48)
\end{aligned}$$

Combining (45),(46),(47) and (48) we have

$$\begin{aligned}
P_{int} = \frac{5}{\pi a^2 \sin \frac{\pi}{5} (2 \cos \frac{\pi}{5} + 1)^2} \left\{ \left(1 - \cos \frac{\pi}{5} \right) \left(15 \cos^2 \frac{\pi}{5} + \right. \right. \\
\left. 15 \cos \frac{\pi}{5} + \frac{9}{2} + \frac{1}{2 \cos \frac{\pi}{5}} \right) - \frac{l^2}{2 \sin \frac{2\pi}{5}} \left[\frac{1}{2} \sin \frac{2\pi}{5} \left(\sin \frac{2\pi}{5} - \right. \right. \\
\left. \left. \cos \frac{2\pi}{5} + 2 \cos \frac{\pi}{5} + 3 \right) + \frac{\pi}{5} \left(\sin \frac{2\pi}{5} + 2 \right) \right] \right\}.
\end{aligned}$$

References

- [1] D. Barilla, G. Caristi and E. Saitta M. Stoka, A Buffon type problem for lattices with central symmetric cell, International Mathematical Forum, vol. 8, 2013, n. 23, 1119-1129.
- [2] G. Caristi and M. Stoka, Bufon-Laplace type problems for three regular lattice and "body test" a parallelogram, International Journal Contemporary Mathematical Forum, vol. 8, 2013, n. 6, pp. 245-271.
- [3] G. Caristi and M. Stoka, A Buffon type problem for an regular lattice I, Rend. Circ. Mat. di Palermo Suppl., Serie II, n. 83, 2011, pp. 65-82.
- [4] G. Caristi and M. Stoka, A Buffon type problem for an regular lattice II, Rend. Circ. Mat. di Palermo Suppl., Serie II, n. 83, 2011, pp. 83-94.
- [5] H. Poincarè, Calcul des probabilités, 2nd ed., Gauthier-Villard, Paris, 1912.
- [6] M. Stoka, Probabilités géométriques de type "Buffon" dans le plan euclidien, Atti Accd. Sci. Torino, T, 110.

Received: July 1, 2013