Asymptotic Properties of Polar Polynomials

Abdelhamid Rehouma
Département de Mathématiques
Université El oued, Algerie

Yamina Laskri
Département de Mathématiques
Université Badji Mokhtar Annaba, Algerie

Rachid Benzine
Département de Mathématiques
Université Badji Mokhtar Annaba, Algerie

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Abstract

Let $\mu$ be a finite positive measure defined on the Borelian $\sigma$–algebra of $\mathbb{C}$, $\mu$ is absolutely continuous with respect to the Lebesgue measure $d\theta$ on $[-\pi, +\pi]$. Let us consider $\{L_n(z)\}_{n \in \mathbb{N}}$, the system of monic orthogonal polynomial with respect to $\mu$. We introduce a new class of polynomials $\{P_n\}$, that we call polar polynomials associated to $\{L_n(z)\}_{n \in \mathbb{N}}$.

For a fixed complex number $\alpha$, $P_n(z)$ is solution of the following differential equation

$$(n + 1) L_n(z) = P_n(z) + (z - \alpha) P'_n(z).$$

we study algebraic and asymptotic properties of the polar polynomials $\{P_n\}_{n \in \mathbb{N}}$.

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1. INTRODUCTION

Let $\mu$ be a finite positive measure defined on the Borelian $\sigma-$algebra of $\mathbb{C}$ and concentrated on the unit circle $T = \{z, |z| = 1\}$. $\mu$ is absolutely continuous with respect to the Lebesgue measure $d\theta$ on $[-\pi, +\pi]$, i.e.

$$d\mu(\theta) = \rho(\theta)d\theta, \quad \rho \geq 0, \quad \rho \in L^1([-\pi, +\pi], d\theta). \quad (1)$$

Let us consider $L_n(z) = z^n + \ldots$, the $n$-th monic (i.e. its leading coefficient is equal to one) a monic orthogonal polynomial with respect to $\mu$, that is

$$\int_{-\pi}^{\pi} L_n(z)(\overline{z})^k \rho(\theta)d\theta = 0, \quad k = 0, 1, \ldots, n - 1, \quad z = e^{i\theta}. \quad (2)$$

For a fixed complex number $\alpha$, that we are going to call the pole, let us define the $P_n(z) = P_{\alpha, n}(z)$ as a monic polynomial, such that

$$(n+1)L_n(z) = ((z - \alpha)P_n(z))' = P_n(z) + (z - \alpha)P_n'(z). \quad (3)$$

$P_n$ is called the $n-th$ polar polynomial of $L_n(z)$. Obviously, $P_n$ is a monic polynomial of degree $n$.

Note that $\Phi_n(z) = (z - \alpha)P_n(z)$ is a monic polynomial, primitive of $(n+1)L_n(z)$, normalized by $\Phi_n(\alpha) = 0$ (see (3)). A direct consequence of (2) and (3) is that $P_n(z)$ satisfy the orthogonality relations

$$\int_{0}^{2\pi} [P_n(z) + (z - \alpha)P_n'(z)](\overline{z})^k \rho(\theta)d\theta = 0, \quad k = 0, 1, \ldots, n - 1, \quad z = e^{i\theta}. \quad (4)$$

This type of orthogonality relations generated by differential operators was introduced initially by Aptekarev, Lagomazino, Marcelan, where the existence and uniqueness conditions for more general differential expressions were studied in detail. A similar study has been done by A. Fundora, H. Pijeira and W. Urbina ([4]), in the case of $[-1, +1]$, instead of the unit circle $T = \{z, |z| = 1\}$.

In this paper we study algebraic and asymptotic properties of the sequence of the polar polynomials $\{P_n\}_{n \in \mathbb{N}}$.

2. AUXILIARY RESULTS

In order to obtain the asymptotic behaviour of the sequence $\{P_n\}_{n \in \mathbb{N}}$, we need some definitions and general results that we will discuss in what follows

**Definition 1.** If $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of measures on a compact set, we say that $\mu_n$ converges weakly to the measure $\mu$ a $n \to \infty$ if

$$\lim_{n \to \infty} \int f d\mu_n = \int f d\mu$$
for every continuous function \( f \) on \( \mathbb{C} \) having compact support. In this case, we write \( \mu_n \overset{x}{\to} \mu \) or \( d\mu_n \overset{x}{\to} d\mu \), or if \( \mu \) is absolutely continuous, \( d\mu_n(x) \overset{x}{\to} \mu'(x)dx \).

**Definition 2.** Let \( Q_n \) be a polynomial of degree exactly \( n \) with simple zeros \( z_{n,1}, z_{n,2}, ..., z_{n,n} \). The normalised counting measure of the zeros of \( Q_n \) is defined by

\[
\nu_n(Q_n) = \frac{1}{n} \sum_{k=1}^{n} \delta_{z_{n,k}}
\]

where \( \delta_{z_{n,k}} \) is the Dirac measure with mass one at the point \( z_{n,k} \).

Let \( \| \cdot \|_\Delta \) denotes the supremum norm on \( \Delta \) and \( \text{Cap}(\Delta) \) the logarithmic capacity of a set \( \Delta \). This being, we will need the following result

**Lemma 1.** Let \( \Delta \subset \mathbb{C} \) be a compact set with empty interior, connected complement and positive logarithmic capacity. If \( \{Q_n(z)\}_{n=0}^{\infty} \) is a sequence of monic polynomials, \( \deg Q_n(z) = n \), such that

\[
\limsup_{n \to \infty} \left\| Q_n \right\|_\Delta \leq \text{Cap}(\Delta)
\]

then

\[
\nu_n(Q_n) \overset{x}{\to} \omega_\Delta
\]

where \( \omega_\Delta \) is the equilibrium measure of \( \Delta \) (see [1], for more details on the notion: equilibrium measure.)

**Lemma 2.** Let \( \{Q_n(z)\}_{n=0}^{\infty} \) be a sequence of polynomials. Then for all \( j \in \mathbb{Z}_+ \)

\[
\limsup_{n \to \infty} \left( \left\| Q_n^{(j)} \right\|_\Delta \right)^{\frac{1}{n}} \leq 1
\]

3. **Asymptotic Formulas**

3.1. **Asymptotics of Derivatives of Orthogonal Polynomials.** Asymptotics for derivatives of orthogonal polynomials have been established under various hypothesis (see [2], [3]).

**Definition 3.** Let \( \mu \) a finite Borel measure on the unit circle (or \( [0, 2\pi] \)). We say that \( \mu \) belongs to the Nevai class \( \mathcal{N} \) (we denote \( \mu \in \mathcal{N} \)) if \( \mu \) verifies the Szegö condition

\[
\int_0^{2\pi} \log(\mu'(\theta))d\theta > -\infty
\]

for some \( \theta \in (0, 2\pi) \) and \( \delta > 0 \). \( \mu \) is absolutely continuous in \((\theta - \delta, \theta + \delta)\), with

\[
\int_{\theta-\delta}^{\theta+\delta} \left( \frac{\mu'(t) - \mu'(\theta)}{t - \theta} \right)^2 dt < \infty
\]
Nevai ([1],[2]) proved the following result

**Theorem 1.** ([6]). Let \( \mu \) a finite Borel measure on the unit circle (or \([0,2\pi]\)). Let \( \{\varphi_n\}_{n \in \mathbb{N}} \) denote the orthonormal polynomials for \( \mu \), so that

\[
\frac{1}{2\pi} \int_0^{2\pi} \varphi_n(e^{i\theta})\overline{\varphi_m(e^{i\theta})}d\mu(\theta) = \delta_{mn}
\]

We suppose that \( \mu \) belongs to the Nevai class \( \mathcal{N} \). Let \( m \geq 1 \). Then

\[
\lim_{n \to \infty} z^m \varphi_n^{(m)}(z) z^{m} \varphi_n(z) = 1, \quad z = e^{i\theta}.
\]

Nevai’s proof involved similar techniques to those for proving the asymptotics for the reversed polynomials

\[
\varphi_n^*(z) = \overline{z^n \varphi_n(\frac{1}{z})}.
\]

By reworking Nevai’s ideas, Lubinsky shows that uniform asymptotics of \( \varphi_n^* \) on an arc directly imply asymptotics for derivatives of \( \varphi_n \). He obtains the following result

**Theorem 2.** ([2]). Let \( J \) be a subinterval of \([0,2\pi]\) and assume that

\[
\lim_{n \to \infty} \varphi_n^*(e^{i\theta}) = g(\theta)
\]

uniformly for \( \theta \in J \), where \( g(\theta) \neq 0 \) for \( \theta \in J \). Let \( m \geq 1 \) and \( I \subset J^\circ \) be a closed interval. Then uniformly for \( z = e^{i\theta}, \theta \in I \)

\[
\lim_{n \to \infty} \frac{z^m \varphi_n^{(m)}(z)}{n^m \varphi_n(z)} = 1. \quad (9)
\]

### 3.2. Asymptotics of Derivatives of monic Orthogonal Polynomials.

Let us consider in this section the set \( \{L_n(z)\}_{n \in \mathbb{N}} \) \( (L_n(z) = z^n + \ldots) \) of monic orthogonal polynomials associated to the measure \( \mu \) \( (d\mu(\theta) = \rho(\theta)d\theta, \text{ see (1.1)}) \). The ratio asymptotic takes the form

\[
\lim_{n \to \infty} \frac{L_{n+1}(z)}{L_n(z)} = z. \quad (10)
\]

There is a major theory for asymptotics of this type, with key initial advances due to Maté, Nevai, Rakhmanov, Totik and many later works (see [2]). In particular, Rakhmanov’s theorem asserts that if \( \mu' > 0 \text{ a.e. on } [0,2\pi] \), then we have this ratio asymptotic. Lubinsky ([2]) proved the following result

**Theorem 3.** ([2]). Assume that

\[
\lim_{n \to \infty} \frac{L_{n+1}(z)}{zL_n(z)} = 1 \quad (11)
\]
holds at some $z$ with $|z| = 1$. Let $m \geq 1$. Then uniformly in $\{ z : |z| \geq 1 \}$,

$$
\lim_{n \to \infty} \frac{L_{n+1}^{(m)}(z)}{z L_n^{(m)}(z)} = 1
$$

(12)

### 3.3. Relative asymptotic of the Polar polynomials $\{P_n\}$

The foregoing preparatory results enable us to study the relative asymptotic of the Polar polynomials $\{P_n\}$ with respect to the monic orthogonal polynomials $\{L_n\}$. Our main result is the following theorem

**Theorem 4.** Let $\alpha$ be a fixed complex number. Let $P_n$ be the polar polynomial of $L_n$. Assume that

$$
\lim_{n \to \infty} L_{n+1}(z) = z
$$

uniformly on compact subsets of the set $\{ z : |z| > 1 \}$. Then we have

$$
\lim_{n \to \infty} \frac{L_n(z)}{P_n(z)} = 1 - \frac{\alpha}{z}
$$

(13)

and

$$
\lim_{n \to \infty} \frac{P_{n+1}(z)}{P_n(z)} = z
$$

(14)

and

$$
\lim_{n \to \infty} \frac{P_n'(z)}{P_n(z)} = z
$$

(14’)

uniformly on compact subsets of the set $\{ z : |z| > 1 \}$

**Proof.** Notice that

$$(n + 1) L_n(z) = P_n(z) + (z - \alpha) P_n'(z).$$

Then

$$
\frac{L_n(z)}{P_n(z)} = (z - \alpha) \frac{P_n'(z)}{n P_n(z)} = (z - \alpha) M(z)
$$

(15)

where

$$
M(z) = \lim_{n \to \infty} \frac{P_n'(z)}{n P_n(z)}.
$$

Next we show the existence of $M(z)$. From the definition of polar polynomials $P_n$ we get

$$
\frac{(n + 1) L_n(z)}{(z - \alpha) P_n(z)} = \frac{1}{z - \alpha} + \frac{P_n'(z)}{P_n(z)} = \frac{1}{z - \alpha} + \frac{d}{dz} \left( \log P_n(z) \right).
$$

The logarithmic derivative and its interpretation depends on location of the zeros of the polynomial $P_n(z)$. 

If we write (see (3))
\[
[(z - \alpha) P_n(z)]' = (n + 1) L_n(z)
\]
then
\[
\frac{[(z - \alpha) P_n(z)]'}{[(z - \alpha) P_n(z)]} = \frac{d}{dz} \log [(z - \alpha) P_n(z)].
\]

hence
\[
\frac{d}{dz} \log [(z - \alpha) P_n(z)] = (n + 1) \frac{L_n(z)}{(z - \alpha) P_n(z)}.
\]

By Integration in both hand sides of this equality from fixed point \( z_1 \) to \( z \), we get,
\[
\frac{(z - \alpha) P_n(z)}{(z_1 - \alpha) P_n(z_1)} = \exp \left[ (n + 1) \int_{z_1}^{z} \frac{L_n(t)}{(t - \alpha) P_n(t)} dt \right],
\]
and
\[
P_n(z) = (z_1 - \alpha) P_n(z_1) \frac{\exp \left[ (n + 1) \int_{z_1}^{z} \frac{L_n(t)}{(t - \alpha) P_n(t)} dt \right]}{z - \alpha}.
\] (16)

(16) implies
\[
\lim_{n \to \infty} \frac{P_{n+1}(z)}{P_n(z)} \frac{P_n(z_1)}{P_{n+1}(z_1)} = \exp \left[ \lim_{n \to \infty} \int_{z_1}^{z} \frac{L_n(t)}{(t - \alpha) P_n(t)} dt \right].
\]

Thus, by considering (15), we have
\[
\lim_{n \to \infty} \frac{P_{n+1}(z)}{P_n(z)} \frac{P_n(z_1)}{P_{n+1}(z_1)} = \exp \left[ \lim_{n \to \infty} \int_{z_1}^{z} \frac{P_n'(t)}{nP_n(t)} dt \right].
\]

Now let \( \Lambda(z) \) be the integral limit
\[
\Lambda(z) = \int_{z_1}^{z} \lim_{n \to \infty} \frac{P_n'(t)}{nP_n(t)} dt = \int_{z_1}^{z} M(t) dt.
\] (17)

Then
\[
\lim_{n \to \infty} \frac{P_{n+1}(z)}{P_n(z)} \frac{P_n(z_1)}{P_{n+1}(z_1)} = e^{\Lambda(z)},
\]

It is easy to conclude that
\[
\lim_{n \to \infty} \frac{P_{n+1}(z)}{P_n(z)} = z_1 e^{\Lambda(z)}
\] (18)

and
\[
e^{\Lambda(z)} = \frac{z}{z_1}.
\] (19)

\( z_1 \) is a fixed point.

(17) and (19) imply
\[ M(z) = \frac{d}{dz} \Lambda(z) = e^{-\Lambda(z)} \frac{1}{z_1} = \frac{1}{z}, \quad (20) \]

(20), (18) and (15) imply
\[
\lim_{n \to \infty} L_n(z) = \frac{z - \alpha}{z} \quad \text{and} \quad \lim_{n \to \infty} \frac{P_{n+1}(z)}{P_n(z)} = z \quad (21)
\]

uniformly on compact subsets of the set \( \{ z : |z| > 1 \} \).

(12) and (14) imply (14').

\[ \square \]

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