The Neutrix Convolution Involving
the Functions $x^r$ and $(1 + x)^s \ln(1 + x_+)$

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Abstract

The neutrix convolutions $(1+x)^s \ln(1+x_+) \ast x^r$ and $x^s \ln(1+x_+) \ast x^r$
are evaluated for $r, s = 0, 1, 2, \ldots$. Further results are also given.

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In the following, $\mathcal{D}$ denotes the space of infinitely differentiable functions
with compact support and $\mathcal{D}'$ denotes the space of distributions defined on $\mathcal{D}$.

The convolution of certain pairs of distributions in $\mathcal{D}'$ is usually defined as
follows, see for example Gel’fand and Shilov [6].

Definition 1. Let $f$ and $g$ be distributions in $\mathcal{D}'$ satisfying either of the
following conditions:
(a) either $f$ or $g$ has bounded support,
(b) the supports of \( f \) and \( g \) are bounded on the same side. Then the convolution \( f \ast g \) is defined by the equation

\[
\langle (f \ast g)(x), \varphi(x) \rangle = \langle g(x), \langle f(t), \varphi(x + t) \rangle \rangle
\]

for arbitrary test function \( \varphi \) in \( \mathcal{D} \).

The classical definition of the convolution is as follows:

**Definition 2.** If \( f \) and \( g \) are locally summable functions then the convolution \( f \ast g \) is defined by

\[
(f \ast g)(x) = \int_{-\infty}^{\infty} f(t)g(x - t) \, dt = \int_{-\infty}^{\infty} f(x - t)g(t) \, dt
\]

for all \( x \) for which the integrals exist.

Note that if \( f \) and \( g \) are locally summable functions satisfying either of the conditions (a) or (b) in Definition 1, then Definition 1 is in agreement with Definition 2.

Definition 1 is rather restrictive and so a neutrix convolution was introduced in [2]. In order to define the neutrix convolution, we first of all let \( \tau \) be the function in \( \mathcal{D} \), see Jones [7], satisfying the following conditions:

(i) \( \tau(x) = \tau(-x) \),
(ii) \( 0 \leq \tau(x) \leq 1 \),
(iii) \( \tau(x) = 1, |x| \leq \frac{1}{2} \),
(iv) \( \tau(x) = 0, |x| \geq 1 \).

The function \( \tau_n \) is now defined by

\[
\tau_n(x) = \begin{cases} 
1, & |x| \leq n, \\
\tau(nx - n^{n+1}), & x > n, \\
\tau(nx + n^{n+1}), & x < -n,
\end{cases}
\]

**Definition 3.** Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \) and let \( f_n = f \tau_n \) for \( n = 1, 2, \ldots \). Then the neutrix convolution \( f \odot g \) is defined to be the neutrix limit of the sequence \( \{f_n \ast g\} \), provided the limit \( h \) exists in the sense that

\[
N \lim_{n \to \infty} \langle f_n \ast g, \varphi \rangle = \langle h, \varphi \rangle
\]

for all \( \varphi \) in \( \mathcal{D} \), where \( N \) is the neutrix, see van der Corput [1], having domain \( N' = \{1, 2, \ldots, n, \ldots\} \) and range the real numbers with negligible functions finite linear sums of the functions

\[
n^\lambda \ln^{r-1} n, \quad \ln^r n \quad (\lambda > 0, \ r = 1, 2, \ldots)
\]

and all functions which converge to zero as \( n \) tends to infinity.
Note that the convolution $f_n \ast g$ in this definition is in the sense of Definition 2, the support of $f_n$ being bounded. Note also that the neutrix convolution in this definition, is in general non-commutative. The convolution $f \ast g$ in the sense of Definition 2 is of course commutative.

It was proved in [2] that if the convolution $f \ast g$ exists by Definition 1, then the neutrix convolution $f \ast \circ g$ exists and

$$f \ast g = f \ast \circ g,$$

showing that Definition 3 is a generalization of Definition 1.

We need the following results which were proved in [3]:

$$\begin{align*}
(1 + x)^r \ln (1 + x_+) \ast x_+^r &= \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \left\{ \frac{(1 + x)^{r+s+1} \ln(1 + x_+)}{r + s - i + 1} 
- \frac{[H(x) + x_+]^{r+s+1} - [H(x) + x_+]^i}{(r + s - i + 1)^2} \right\} 
\tag{1}
\end{align*}$$

for $r, s = 0, 1, 2, \ldots$, where $H(x)$ denotes Heaviside’s function.

$$\begin{align*}
(1 - x)^r \ln (1 + x_-) \ast x_-^r &= \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \left\{ \frac{(1 - x)^{r+s+1} \ln(1 + x_-)}{r + s - i + 1} 
- \frac{[H(-x) + x_-]^{r+s+1} - [H(-x) + x_-]^i}{(r + s - i + 1)^2} \right\} 
\tag{2}
\end{align*}$$

for $r, s = 0, 1, 2, \ldots$.

$$\begin{align*}
x^r \ln (1 + x_+) \ast x_+^r &= \sum_{i=0}^{r} \binom{r}{i} \sum_{j=0}^{s} \binom{s}{j} (-1)^{s-j-r-i} \left\{ \frac{(1 + x)^{r+j+1} \ln(1 + x_+)}{r + j - i + 1} 
- \frac{[H(x) + x_+]^{r+j+1} - [H(x) + x_+]^i}{(r + j - i + 1)^2} \right\} 
\tag{3}
\end{align*}$$

for $r, s = 0, 1, 2, \ldots$.

$$\begin{align*}
x^r \ln (1 + x_-) \ast x_-^r &= \sum_{i=0}^{r} \binom{r}{i} \sum_{j=0}^{s} \binom{s}{j} (-1)^{r+s-j-i} \left\{ \frac{(1 - x)^{r+j+1} \ln(1 + x_-)}{r + j - i + 1} 
- \frac{[H(-x) + x_-]^{r+j+1} - [H(-x) + x_-]^i}{(r + j - i + 1)^2} \right\} 
\tag{4}
\end{align*}$$

for $r, s = 0, 1, 2, \ldots$.
Since the neutrix convolution is not necessarily commutative, we now prove

**Theorem 1.** The neutrix convolution $x^r \odot (1 + x)^s \ln(1 + x_+)$ exists and

$$x^r \odot (1 + x)^s \ln(1 + x_+) = \sum_{i=0}^{r} \sum_{k=1}^{r-s-i+1} \binom{r}{i} \binom{r+s-i+1}{k} \frac{(1 + x)^{i+k}(-1)^{r+i+k+1}}{k(r+s-i+1)},$$

(5)

for $r, s = 0, 1, 2, \ldots$

**Proof.** Putting $[x^n]_n = x^n \tau_n(x)$ and $u = 1 + t$, we have

$$[x^n]_n \ast (1 + x)^s \ln(1 + x_+) = \int_0^{x^n} (1 + t)^s \ln(1 + t)(x - t)^r dt$$

$$+ \int_{x^n}^{x^n+1} (1 + t)^s \ln(1 + t)(x - t)^r \tau_n(t) dt$$

$$= \int_1^{x^n+1} u^s \ln u(1 + u)^r dt$$

$$+ \int_{x^n}^{x^n+1} (1 + t)^s \ln(1 + t)(x - t)^r \tau_n(t) dt$$

$$= I_1 + I_2,$$

(6)

where

$$I_1 = \sum_{i=0}^{r} \binom{r}{i} (1 + x)^i (-1)^{r-i} \frac{(x + n + 1)^{r+s-i+1}}{r + s - i + 1}$$

$$- \frac{(x + 1 + n)^{r+s-i+1} - 1}{(r + s - i + 1)^2}.$$  

(7)

It follows that

$$\lim_{n \to \infty} I_1 = \lim_{n \to \infty} \sum_{i=0}^{r} \binom{r}{i} \binom{r+s-i+1}{k} \frac{(-1)^{j+1}n^{k-j}(1 + x)^j}{j} + n^k \ln n$$

$$= \sum_{i=0}^{r} \sum_{k=1}^{r+s-i+1} \binom{r}{i} \binom{r+s-i+1}{k} \frac{(1 + x)^{i+k}(-1)^{r+i+k+1}}{k(r+s-i+1)}.$$  

(7)

Next, since $I_2 = O(n^{-n})$, it follows that

$$\lim_{n \to \infty} I_2 = 0.$$  

(8)
Equation (5) now follows from equations (6) to (8).

**Corollary 1.1** The neutrix convolution \( x^r \otimes (1 - x)^s \ln(1 + x_-) \) exists and

\[
x^r \otimes (1 - x)^s \ln(1 + x_-) = \sum_{i=0}^{r} \sum_{k=1}^{r+s-i+1} \binom{r}{i} \binom{r+s-i+1}{k} \frac{(1-x)^{i+k}(-1)^{i+k+1}}{k(r+s-i+1)},
\]

for \( r, s = 0, 1, 2, \ldots \).

**Proof.** Equation (9) follows from equation (5) on replacing \( x \) by \(-x\).

**Corollary 1.2** The neutrix convolution \( x_-^r \otimes (1 + x)^s \ln(1 + x_+) \) exists and

\[
x_-^r \otimes (1 + x)^s \ln(1 + x_+) = \sum_{i=0}^{r} \sum_{k=1}^{r+s-i+1} \binom{r}{i} \binom{r+s-i+1}{k} \frac{(1+x)^{i+k}(-1)^{i+k}}{k(r+s-i+1)}
\]

\[
- \sum_{i=0}^{r} \binom{r}{i} (-1)^i \left\{ \frac{(1+x)^{r+s+1} \ln(1 + x_+)}{r+s-i+1} - \frac{[H(x) + x_+]^{r+s+1} - [H(x) + x_+]^i}{(r+s-i+1)^2} \right\}
\]

for \( r, s = 0, 1, 2, \ldots \).

**Proof.** Using equations (1) and (5), we have

\[
x^r \otimes (1 + x)^s \ln(1 + x_+) = [x_+^r + (-1)^r x_-^r] \otimes (1 + x)^s \ln(1 + x_+)
\]

\[
= \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \left\{ \frac{(1+x)^{r+s+1} \ln(1 + x_+)}{r+s-i+1} - \frac{[H(x) + x_+]^{r+s+1} - [H(x) + x_+]^i}{(r+s-i+1)^2} \right\}
\]

and equation (10) follows.

**Corollary 1.3** The neutrix convolution \( x_+^r \otimes (1 - x)^s \ln(1 + x_-) \) exists and

\[
x_+^r \otimes (1 - x)^s \ln(1 + x_-) = \sum_{i=0}^{r} \sum_{k=1}^{r+s-i+1} \binom{r}{i} \binom{r+s-i+1}{k} \frac{(1-x)^{i+k}(-1)^{i+k}}{k(r+s-i+1)}
\]

\[
- \sum_{i=0}^{r} \binom{r}{i} (-1)^i \left\{ \frac{(1-x)^{r+s+1} \ln(1 + x_-)}{r+s-i+1} - \frac{[H(-x) + x_-]^{r+s+1} - [H(-x) + x_-]^i}{(r+s-i+1)^2} \right\}
\]

(11)
for \( r, s = 0, 1, 2, \ldots \).

**Proof.** Equation (11) follows from equation (10) on replacing \( x \) by \(-x\).

**Theorem 2.** The neutrix convolution \( x^r \otimes x^s \ln(1 + x) \) exists and

\[
x^r \otimes x^s \ln(1 + x) = \sum_{i=0}^{r} \sum_{j=0}^{s} \sum_{k=1}^{r+j-i+1} \binom{r}{i} \binom{s}{j} \frac{(r + k - i + 1)}{k} \frac{(-1)^{s-j+r-i+k+1}(1+x)^{i+k}}{k(r+j-i+1)}
\]

(12)

for \( r, s = 0, 1, 2, \ldots \).

**Proof.** Putting \([x^r]_n = x^r \tau_n(x)\) and \( u = 1 + t \), we have

\[
[x^r]_n \otimes x^s \ln(1 + x) = \int_0^{x+n} t^s \ln(1+t)(x-t)^r dt
\]

\[
+ \int_{x+n}^{x+n+n^+} t^s \ln(1+t)(x-t)^r \tau_n(t) dt
\]

\[
= \int_{1+x}^{x+n+1} (u-1)^s \ln u(1+u-x)^r dt
\]

\[
+ \int_{x+n}^{x+n+n^+} t^s \ln(1+t)(x-t)^r \tau_n(t) dt
\]

\[
= J_1 + J_2,
\]

(13)

where

\[
J_1 = \sum_{i=0}^{r} \binom{r}{i} \sum_{j=0}^{s} \binom{s}{j} (-1)^{s-j+r-i}(1+x)^i \int_1^{x+n+1} u^{r+j-i} \ln u du
\]

\[
= \sum_{i=0}^{r} \binom{r}{i} \sum_{j=0}^{s} \binom{s}{j} (-1)^{s-j+r-i}(1+x)^i \left[ \frac{(x+n+1)^{r+j-i+1} \ln(x+n+1)}{r+j-i+1} \right]
\]

\[
- \frac{(x+n+1)^{r+j-i+1} - 1}{(r+j-i+1)^2}
\]

It follows that

\[
\text{N-} \lim_{n \to \infty} J_1 = \text{N-} \lim_{n \to \infty} \sum_{i=0}^{r} \sum_{j=0}^{s} \binom{r}{i} \binom{s}{j} \frac{(1+x)^i(-1)^{s-j+r-i}}{r+j-i+1}
\]

\[
\times \sum_{k=0}^{r+j-i+1} \binom{r+j-i+1}{k} \left[ \sum_{m=1}^{\infty} \frac{(-1)^{m+1} n^{k-m}(1+x)^j}{m} \right] + n^k \ln n
\]

\[
= \sum_{i=0}^{r} \sum_{j=0}^{s} \sum_{k=1}^{r+j-i+1} \binom{r}{i} \binom{s}{j} \frac{(r+k-i+1)}{k} \frac{(-1)^{s-j+r-i+k+1}(1+x)^{i+k}}{k(r+j-i+1)}
\]

(14)
It follows as above that
\[
\lim_{n \to \infty} J_2 = 0 \quad (15)
\]
and equation (12) now follows from equations (13) to (15).

**Corollary 2.1** The neutrix convolution \( x^r \otimes x^s \ln(1 + x_-) \) exists and

\[
x^r \otimes x^s \ln(1 + x_-) = \sum_{i=0}^{r} \sum_{j=0}^{s} \sum_{k=1}^{r+j-i+1} \binom{r}{i} \binom{s}{j} \frac{(1-x)^{i+k}(-1)^{j-i+k+1}}{k(r+j-i+1)} \quad (16)
\]
for \( r, s = 0, 1, 2, \ldots \).

**Proof.** Equation (16) follows from equation (12) on replacing \( x \) by \(-x\).

**Corollary 2.2** The neutrix convolution \( x^r_- \otimes x^s \ln(1 + x_+) \) exists and

\[
x^r_- \otimes x^s \ln(1 + x_+) = \sum_{i=0}^{r} \sum_{j=0}^{s} \sum_{k=1}^{r+j-i+1} \binom{r}{i} \binom{s}{j} \left( \frac{(1-x)^{i+k}(-1)^{j-i+k+1}}{k(r+j-i+1)} - \frac{[H(x) + x_+]^{r+j+1} - [H(x) + x_+]^{i}}{(r+j-i+1)^2} \right) \quad (17)
\]
for \( r, s = 0, 1, 2, \ldots \).

**Proof.** Using equations (3) and (12), we have

\[
x^r \otimes x^s \ln(1 + x_+) = [x^r_+ + (-1)^rx_-^r] \otimes x^s \ln(1 + x_+)
\]

\[
= \sum_{i=0}^{r} \binom{r}{i} \sum_{j=0}^{s} (-1)^{s-j-r-i} \left( \frac{(1-x)^{r+j+1} \ln(1+x_+)}{r+j-i+1} - \frac{[H(x) + x_+]^{r+j+1} - [H(x) + x_+]^{i}}{(r+j-i+1)^2} \right) + (-1)^r x^r_- \otimes x^s \ln(1 + x_+)
\]
and equation (17) follows.
Corollary 2.3 The neutrix convolution $x_+^r \otimes x^s \ln(1 + x_-)$ exists and

$$x_+^r \otimes x^s \ln(1 + x_-) = \sum_{i=0}^{r} \sum_{j=0}^{s} \sum_{k=1}^{r+j-i+1} \binom{r}{i} \binom{s}{j} \binom{r+k-i+1}{k}$$

$$\times \frac{(-1)^{j-i+k+1}(1-x)^{i+k}}{k(r+j-i+1)}$$

$$-\sum_{i=0}^{r} \binom{r}{i} \sum_{j=0}^{s} \binom{s}{j} (-1)^{j-i} \left[ \frac{(1-x)^{r+j+1} \ln(1 + x_-)}{r+j-i+1} \right]$$

$$- \frac{[H(-x) + x_-]^{r+j+1} - [H(-x) + x_-]^i}{(r+j-i+1)^2}, \quad (18)$$

for $r, s = 0, 1, 2, \ldots$.

**Proof.** Equation (18) follows from equation (17) on replacing $x$ by $-x$.

For further results on the neutrix convolution, see [4] and [5].

**References**


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