Geometrically Nonlinear Deformation

Elastoplastic Soil

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Abstract

In work statement of a problem of numerical modeling of finite deformations elastoplastic soil environments, focused on application FEM is given. Implemented and tested method of solving the problem of elastic-plastic deformation of soil masses on the basis of defining relations between the increments of the true stress and strain, resolved a number of model problems of determining the stress-strain and limiting condition of soils.

Keywords: a method of finite elements, finite strains, elastoplastic soil

1. Introduction

The main direction of a number of challenges facing soil mechanics is the theoretical prediction of the behavior of groundwater strata under the influence of
internal and external influences: a variety of loads from the structures, the changes under the influence of natural factors and human activities, the conditions of equilibrium structures, such as washouts, water-table fluctuations and unloading of deep soil layers during digging of foundation pits, etc.

The research problem of a stress-strain condition of soils under the influence of external forces and a body weight is the basic in the mechanics of soils, and its solving for various load cases has the direct appendix in building practice. In building it is rather important to know, how stress in a ground are distributed at loading of a part of its surface, under what conditions there comes a limiting intense condition then there are inadmissible deformations and discontinuity of a soil mass, etc. Important role plays the mathematical modeling, allowing predicting and optimising technological influences, to process and interpreting experimental data.

Traditionally, solid mechanics for solving geometrically nonlinear problems has spread Lagrangian description of the environment in which the well-formulated boundary value problem in differential or variational forms, the solution of which can use a variety of numerical algorithms [3, 4]. Within the framework of modern numerical methods have been developed step by step methods of loading, in accordance with which the deformation is represented as a sequence of equilibrium states, and the transition from the current state to the subsequent increments determined load change in the boundary conditions or the computational domain, etc.

In this paper we give a formulation of the problem of numerical modeling of finite deformations of elastic-plastic soil environments focused on the application of the finite element method. Implemented and tested method of solving the problem of elastic-plastic deformation of soil masses on the basis of defining relations between the increments of the true stress and strain. Solved a number of model problems of determining the stress-strain and limiting state soil.

2. The algorithm for calculating the nonlinear deformation of elastic-plastic soil

2.1. Common relations

Let's take advantage of the statement resulted in [5, 6]. Let deformable solid body has a volume, \( V_0 \) is limited by a surface \( S_0 \), is related to the orthogonal Cartesian coordinate system \( x' \) \((i=1,2,3)\). Position of an arbitrary point \( M \) before deformation we will define a radius-vector \( R = x' \hat{e}_i \), and after deformation – a radius-vector \( R = R + \bar{u} = (x' + u')\hat{e}_i \), where \( \hat{e}_i \) the unit vectors introduced spatial orthogonal coordinate system, \( \bar{u} \) - the displacement vector. After deformation the elementary rectangular parallelepiped with the parties \( dl_i = dx' \) becomes in a curvilinear parallelepiped. Lengths of edges of a curvilinear parallelepiped and the
area of its sides we will designate through \( dV' \) and \( S' \) accordingly. The areas of

\[ S_i = dx^i dx^i, \quad S_e = dx^i dx^e \cdot \]

On the sides of a curvilinear parallelepiped having the areas \( \Sigma \) we will enter into consideration stress vectors \( \sigma' \) carried to units of the areas, \( \Sigma \) and also vectors of volume and superficial forces \( F' \) and \( F' \), carried to units of initial volume \( dV = dx^i dx^e dx^3 \) and the initial area \( dS_0 \). State of static equilibrium of the body is described by the variational equation of the principle of virtual displacements

\[
\int_{V_0} \sigma' \delta u \, dV_0 \ = \ \int_{V_0} F' \delta u \, dV_0 \ + \ \int_{\Sigma} \bar{F}' \delta u \, dS_0. \tag{1.1}
\]

Let’s accept for decomposition \( \sigma'_i \) vectors, \( \sigma'_i = \sigma'_i R'_i = s'_i R'_i \) where

\[ R'_i = \partial R'_j / \partial x' = \delta_i + u_j - \] the basic vectors in the deformed condition of a body the left part of the equation (1.1) will be transformed to a kind

\[
\int_{V_0} \sigma'_i \delta u_i \, dV_0 = \int_{V_0} \sigma'_i \delta u_i \, dV_0 \ . \tag{1.2}
\]

Entering into (1.2) components symmetric tensor \( \varepsilon\)

\[
\varepsilon' = \frac{1}{2} \left( \bar{R}'_i R'_j - R'_i R'_j \right) = \frac{1}{2} \left( \varepsilon'_i - \delta_i \right) = \frac{1}{2} \left( \delta_{ij} + \varepsilon_{ij} \right), \quad \varepsilon' = u'_j = \partial u'_j / \partial x' \quad \tag{1.3}
\]

called the components of the strain tensor Cauchy-Green [1, 3, etc.], and sizes on V.V. Novozhilov \( \sigma'_i \) are called as the generalized stresses. True deformations of lengthening \( \varepsilon \) and shifts \( \sin \gamma \) we call the value [2]

\[
\varepsilon'' = \varepsilon = dV' / dV = -1 - \sqrt{1 + 2 \varepsilon}, \quad \varepsilon'' = \sin \gamma =
\]

\[
2 \varepsilon' \left( 1 + 2 \varepsilon \right)^{-3/2} \left( 1 + 2 \varepsilon' \right)^{-1} = 2 \varepsilon' \left( 1 + \varepsilon \right)^{-3/2} \left( 1 + \varepsilon' \right)^{-1},
\]

and true stress \( \sigma \neq \sigma' \) carried to units of the deformed areas \( \Sigma \), on V.V. Novozhilov is components of vectors \( \sigma \), in representations \( \sigma = \sigma_i e^i \) where

\[ \delta = R'_i / \bar{R}' = \left( 1 + \varepsilon \right)^{-1} \left( \delta_{ij} + \varepsilon_{ij} \right) e^i \] the unit vectors directed on tangents to deformed coordinate lines in a point \( M(x') \) into which the point \( M(x') \) after deformation. Since \( S \sigma = S' \sigma \), between components \( \sigma'_i \) and \( \sigma \) dependences \( \sigma'_i = S' \sigma / S \left( 1 + \varepsilon \right) \) transformed to

\[
\sigma'_{11} = \frac{1}{1 + \varepsilon} \left( 1 + \varepsilon_2 \right) \left( 1 + \varepsilon_3 \right) \cos \gamma_{23}, \quad \sigma'_{22} = \sigma_{22} \left( 1 + \varepsilon_2 \right) \cos \gamma_{23} = \sigma_{23} \left( 1 + \varepsilon_3 \right) \cos \gamma_{23} = \sigma'_{23}; \tag{1.5}
\]

Use according to the true values of the strain and stress (1.4), (1.5) form the expression for the variation of the potential energy of deformation

\[
\delta II = \int_{V_0} \sigma'_i \delta u_i \, dV_0 = \int_{V_0} \left( \tau_i \delta e_i + \tau_{ij} \delta e_j + \tau_{ij} \delta e_i + \tau_{ij} \delta \sin \gamma_{ij} + \right.
\]

\[ + \tau_{ij} \delta \sin \gamma_{ij} + \tau_{ij} \delta \sin \gamma_{ij} \right) (1 + \varepsilon_1) (1 + \varepsilon_2) (1 + \varepsilon_3) dV_0, \tag{1.6}
\]

where we have introduced the notation

\[
\tau_i = \sigma_{11} \cos \gamma_{23} / (1 + \varepsilon_1) + \sigma_{12} \cos \gamma_{13} \sin \gamma_{12} + \sigma_{13} \cos \gamma_{23} \sin \gamma_{23}, \quad \tau_{ij} = \sigma_{ij} \cos \gamma_{23} = \sigma_{ij} \cos \gamma_{23} = \tau_{ij}; \tag{1.5.3}
\]

As a condition of plasticity in the work of the criterion of Huber-Mises yield
function for which $F = \sigma_t - \sigma_r$, where $\sigma_t$—stress intensity, $\sigma_r$—yield stress. For a number of soils limit state is well described by the condition Mises-Botkin, which is written in the form

$$F = \tau_v - c'' + \sigma_t g\phi'',$$

where $\phi''$—angle of internal friction in the octahedral sites, $c''$—ultimate resistance to pure shear. Type equation Prandtl-Reuss associated components increments true stress $\sigma'_0$ and true strain $\varepsilon''_0$

$$\Delta \sigma'_0 = 2G \left( \Delta \varepsilon''_0 + \delta_v - 3\frac{3\mu}{1-2\mu} \Delta \varepsilon''_0 \right) - \alpha \frac{(G/\tau_v) \sigma'_0 + Ktg\phi'' \delta_v \varepsilon''_0}{\frac{G}{(3(1-2\mu))}tg^2\phi''},$$

where $\Delta \varepsilon''_0$—the increment of the average true strain, $\Delta \varepsilon''_0$—increment component of true strain, $\sigma'_0$—components of the stress deviator, $\tau_v$—the shear stress intensity, $\delta_v$—Kronecker delta, $\mu$—Poisson's ratio, $G$—shear modulus, $E$—Young's modulus.

### 2.2. Relations "modified incremental theory of Lagrange"

In this case, the variational equation of the principle of virtual displacements (1.1) can be represented as

$$\int \int \int \sigma'_0 \delta \varepsilon'_0 \delta V_0 = \int \int \int F'_i \delta u'_i dV_0 + \int \int P'_i \delta u'_i dS_0.$$

In accordance with the method used [1, 2], is ideally suited for the solution of problems in the theory of the course and called "modified Lagrangian incremental theory" [1], the deformation is represented as a sequence of equilibrium states at the appropriate levels of loading

$$(1'F'_i, 1'P'_i), (2'F'_i, 2'P'_i), \ldots (l'F'_i, l'P'_i), \ldots$$

It is considered that a known $l$-th state, i.e. found displacement $l'u'$, the components of the strain tensor Cauchy-Green $l\varepsilon'_0$, the true strain $l\varepsilon''_0$ and stress $l\sigma'_0$, satisfy the variational equation of the principle of virtual displacements

$$\delta^l \mathcal{C} = \int \int \int l\varepsilon'_0 \delta \varepsilon'_0 dV - R = 0, \quad \delta^l R = \int \int \int l'F'_i \delta u'_i dV + \int \int \int l'P'_i \delta u'_i dS$$

(2.1)

it follows from the equations of equilibrium

$$f^l_i ('u') + l'F'_i = 0$$

and relevant to them, or static (at "soft" power loading) or kinematic boundary conditions, formulated with "hard" kinematic loading samples. If loadings operating on a body receive increments $\Delta F'_i, \Delta P'_i$ then all points of a body receive increase of displacements $\Delta 'u'$ which must satisfy the variational equation for $(l+1)$th state of the form

$$\delta^{l+1} \mathcal{C} = \int \int \int (l\varepsilon'_0 + \Delta l\varepsilon'_0) \delta \varepsilon'_0 dV - \int \int \int (l'F'_i + \Delta l'F'_i) \delta u'_i dV - \int \int \int (l'P'_i + \Delta l'P'_i) \delta u'_i dS,$$
since for \((l+1)\)th state \(\delta^i u' = 0\), \((i.e \delta^i e_y = 0, \delta^i e_y = 0)\). For transformation of the equation (2.2), parities (1.4) we will present in the form
\[
\sigma^x_n = \sigma_n (1 + f_n (\varepsilon_{an})), \sigma^y_n = \sigma_n (1 + f_n (\varepsilon_{an})),
\]
(2.3)
where
\[
f_{11} = (1 + \varepsilon_n) (1 + \varepsilon_n) \cos \gamma_{23} / (1 + \varepsilon_n) - 1, f_{12} = (1 + \varepsilon_n) \cos \gamma_{23} - 1, \]
(2.3)

At use of parities (2.3) in the left part of the equation (2.2) it is had following transformations
\[
\int \int \int \left( \delta^x_n + \Delta^x_n \right) \delta \Delta^x_n e_y dV = \int \int \int \left( \sigma^x_n + \Delta^x_n \sigma^x_n \right) \left( \delta^x_n + \Delta^x_n \delta^x_n \right) \Delta^x_n e_y dV =
\]
\[
= \int \int \int \left( \sigma^x_n + \Delta^x_n \sigma^x_n \right) \left( 1 + f_n + \Delta f_n \right) \left( \delta^x_n + \Delta^x_n \delta^x_n \right) \Delta^x_n e_y dV =
\]
\[
= \int \int \int \left( \sigma^x_n + D_{an} \Delta^x_n e_n + D_{an} \left( \Delta^x_n e_{an}, e_{an} \right) \right) \left( 1 + f_n + \Delta f_n \right) \left( \delta^x_n + \Delta^x_n \delta^x_n \right) \Delta^x_n e_y dV,
\]
where in the underlined expressions there is no summation on indexes \(i\) and \(k\).

Let’s substitute this equality in (2.2) and we will receive the equation
\[
\int \int \int D_{an} \Delta^x_n e_n \Delta^x_n e_y dV + \int \int \int \sigma_n (1 + f_n + \Delta f_n) \Delta^x_n e_y \Delta^x_n e_y dV =
\]
\[
= R + \Delta R - \int \int \int \sigma_n \left( 1 + f_n + \Delta f_n \right) \left( \delta^x_n + \Delta^x_n \delta^x_n \right) \Delta^x_n e_y dV - \int \int \int \sigma_n \Delta f_n \left( \delta^x_n + \Delta^x_n \delta^x_n \right) \Delta^x_n e_y dV -
\]
\[
- \int \int \int D_{an} \left( \Delta^x_n e_{an}, e_{an} \right) \left( 1 + f_n + \Delta f_n \right) \left( \delta^x_n + \Delta^x_n \delta^x_n \right) \Delta^x_n e_y dV - \int \int \int D_{an} \Delta^x_n e_n \left( \delta^x_n + \Delta^x_n \delta^x_n \right) \Delta^x_n e_y dV -
\]
\[
= R + \Delta R - \int \int \int \left( \sigma_n (1 + f_n) \left( \delta^x_n + \Delta^x_n \delta^x_n \right) \Delta^x_n e_y dV \right) - \int \int \int \sigma_n \Delta f_n \left( \delta^x_n + \Delta^x_n \delta^x_n \right) \Delta^x_n e_y dV - A^x \left( e_{an}, \Delta^x_n e_{an} \right),
\]
(2.4)

In the right part of the given equation the second composed the left part and the first with the third composed the right part represent the variation equation (2.1) for \(i\) th state in which instead of a variation \(\delta^i u'\) possible displacements are variations of increments \(\Delta^i u'\). Since at the \(l\)-th step of loading the equation (2.1) with algorithms performed with some error, the governing equation (2.4), these terms are usually stored, which provides greater accuracy of the solution of problems. Other equations composed to the right part (2.4) with factors on \(\sigma_n\) the first step of loading are equal to zero because \(\sigma_n = 0\), and the terms, united under the operator \(A^x \left( e_{an}, \Delta^x_n e_{an} \right)\) under the assumption that the increments of displacement step of loading the values are small compared with the rest. When pruning at each step of loading a system of linear equations to determine \(\Delta^i u'\) in the form
\[
\int \int \int D_{an} \Delta^x_n e_n \Delta^x_n e_y dV + \int \int \int \sigma_n (1 + f_n + \Delta f_n) \Delta^x_n e_y \Delta^x_n e_y dV =
\]
\[
= R + \Delta R - \int \int \int \sigma_n \left( 1 + f_n + \Delta f_n \right) \left( \delta^x_n + \Delta^x_n \delta^x_n \right) \Delta^x_n e_y dV - \int \int \int \sigma_n \Delta f_n \left( \delta^x_n + \Delta^x_n \delta^x_n \right) \Delta^x_n e_y dV,
\]
wherein in a first step of loading \( \sigma_a = 0 \). Transition to a following step load assumes calculations
\[
\Delta u' = u' + \Delta' u', \quad \Delta \sigma_y = D_{xx,xx} \Delta' \varepsilon_{xx} + D_{xx,yy} \Delta' \varepsilon_{yy}, \quad \Delta \sigma_y = \Delta' \sigma_y + B_{yy} \Delta' \sigma_y, \quad \Delta R = \Delta' R,
\]
where \( B_{yy} \Delta' \sigma_y \) - the operator of transition a component of increments of stress from \((l+1)\) th condition in \( l \) th, and stress are written down \( \Delta \sigma_y \) in basis \( l \) th of condition. It is necessary to carry out iterative refinement to take account in the equation (2.4) terms \( \Delta \sigma_y \). Transition to the following iteration assumes following calculations
\[
\Delta' u'_{\text{iter}} = \Delta' u' + \Delta' \sigma_y^{(l+1)} = D_{xx,xx} \Delta' \varepsilon_{xx} + D_{xx,yy} \Delta' \varepsilon_{yy}, \quad \Delta' \sigma_y^{(l+1)} = \Delta' \sigma_y + B_{yy} \Delta' \sigma_y^{(l+1)},
\]
where \( q \) - iteration number. Transition to a following step assumes recalculation of stress in basis \((l+1)\) of the condition.

3. Numerical results

3.1. Elastoplastic calculation of earth embankment

We calculate the earth embankment under its own weight and the load uniformly distributed along the upper boundary of the computational domain, the lower bound has no vertical displacement, and the side - horizontal (Figure 1a). The case of plane strain. It was decided that the ground - a homogeneous medium with the following physical and mechanical properties: Young's modulus \( E = 0.16 \, \text{MPa} \), Poisson's ratio \( \mu = 0.42 \), coupling \( C = 40 \, \text{KPa} \), angle of internal friction \( \varphi = 17 \)°, density \( \rho = 2000 \, \text{kg/m}^3 \), loading \( q = 0.40 \, \text{MPa} \), \( H_1 = 40 \, \text{m} \), \( H_2 = 45 \, \text{m} \), \( L = 211.29 \, \text{m} \), \( h = 14 \, \text{m} \), \( l = 55 \, \text{m} \).

![Fig. 1.](image)

The calculation was based on the bilinear 4-node finite elements and quadratic 8-node elements Sirendipova type. Figure 1b shows the intensity distribution of plastic deformations. Table 1 compares some of the results are carried out of the solution (the intensity of the normal stresses \( \sigma_i [\text{MPa}] \), the maximum deflection \( u_i [\text{m}] \) and intensity of plastic deformations \( \varepsilon_i \) ) obtained by the proposed method and by other authors.
Table 1. Comparison of results of decisions

<table>
<thead>
<tr>
<th>Compared values</th>
<th>4-node element</th>
<th>8-node element</th>
<th>ANSYS Plane42</th>
<th>The solve [132]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_i , [MPa] )</td>
<td>0.593</td>
<td>0.617</td>
<td>0.590</td>
<td>0.588</td>
</tr>
<tr>
<td>( u_i , [m] )</td>
<td>0.854</td>
<td>0.897</td>
<td>0.854</td>
<td>0.854</td>
</tr>
<tr>
<td>( \varepsilon_i )</td>
<td>0.022</td>
<td>0.023</td>
<td>0.022</td>
<td>0.023</td>
</tr>
</tbody>
</table>

3.2. Elastoplastic deformation of soil in the subway tunnel

Calculations are carried out with the soil mass situated therein lining subway tunnel under its own weight, the lower bound has no vertical displacement, and the side - horizontal (Figure 2a). The case of plane strain. It was assumed that the soil can be of two types, mechanical properties which (as the characteristics of the concrete, which is made of lining the tunnel underground) are shown in Table 2. Geometrical parameters of the computational domain (Figure 2a) as follows: \( H = 45 \, m \), \( L = 211.29 \, m \), \( h = 14 \, m \), \( l = 55 \, m \).

Table 2.

<table>
<thead>
<tr>
<th>Environment</th>
<th>Young's modulus</th>
<th>Poisson's ratio</th>
<th>Density</th>
<th>Coupling</th>
<th>Angle of internal friction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sandy loam</td>
<td>12</td>
<td>0.3</td>
<td>2060</td>
<td>15</td>
<td>22</td>
</tr>
<tr>
<td>Sand</td>
<td>33</td>
<td>0.3</td>
<td>2040</td>
<td>2.6</td>
<td>33</td>
</tr>
<tr>
<td>Concrete</td>
<td>35000</td>
<td>0.2</td>
<td>2500</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Numerical calculations indicate reaching the ground state in the limit of the computational domain. This is illustrated in Figure 2b-2c, which shows the area of plastic deformation (in the area of the underground tunnel lining), respectively, for forming the computational domain of soils 1 and 2 (Table 2)

![Fig. 2.](image)
4. Analyzing results and conclusion

In the problem of deformation of a dirt mound, the zones of maximum plastic deformations occur in the lower slopes of a dirt mound, which is usually borne out in practice. In the problem of deformation of soil in the tunnel lining metro area, distribution of plastic strains are identical for both types of soil. Sand (see Fig. 2c) is clearly observed sediment soil (caused by plastic deformation) directly above the tunnel lining, which can be explained by the weaker adhesion of sand compared to the sandy loam.

Analysis of the results shows that the implemented in the method of calculation of the stress-strain and limit state in a physically non-linear soil massif with the finite deformations gives results comparable as to the results obtained in the ANSYS, and with the results of other authors.

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