A Note on the Twisted Changhee Polynomials with $q$-Parameter

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Abstract

In this paper, we consider Changhee polynomials with $q$-parameter and investigate some properties of those polynomials.

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1 Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will respectively denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers and the completions of algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm is defined $|p|_p = \frac{1}{p}$.

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When one talks of $q$-extension, $q$ is various considered as an indeterminate, a complex $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation:

$$[x]_q = \frac{1 - q^x}{1 - q}.$$ 

Note that $\lim_{q \to 1}[x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the $p$-adic invariant integral on $\mathbb{Z}_p$ is defined by by Kim as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{n \to \infty} \frac{1}{p^n} \sum_{x=0}^{p^n-1} f(x) (-1)^x, \text{ (see \cite{7, 8}).} \quad (1.1)$$

Let $f_1$ be the translation of $f$ with $f_1(x) = f(x + 1)$. Then, by (1.1), we get

$$I(f_1) + I(f) = 2f(0). \quad (1.2)$$

As it is known, the Stirling number of the first kind is defined by

$$(x)_n = x(x-1) \cdots (x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \quad (1.3)$$

and the Stirling number of the second kind is given by the generating function to be

$$(e^t - 1)^m = m! \sum_{l=m}^{\infty} S_2(l,m) \frac{t^l}{l!}, \quad (1.4)$$

(see \cite{1, 13}).

Unsigned Stirling numbers of the first kind is given by

$$x^n = x(x+1) \cdots (x+n-1) = \sum_{l=0}^{n} |S_1(n,l)| x^l. \quad (1.5)$$

Note that if we replace $x$ to $-x$ in (1.3), then

$$(-x)_n = (-1)^n x^n = (-1)^n \sum_{l=0}^{n} |S_1(n,l)| x^l$$

$$= \sum_{l=0}^{n} S_1(n,l)(-1)^l x^l. \quad (1.6)$$

Hence $S_1(n,l) = |S_1(n,l)|(-1)^{n-l}$. 
For \( r \in \mathbb{N} \), as is well known, the \textit{Euler polynomials of order} \( r \) are defined by the generating function to be
\[
\left( \frac{2}{e^t + 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}, \quad \text{(see [2, 4, 5, 11])}. \tag{1.7}
\]

When \( x = 0 \), \( E_n^{(r)} = E_n^{(r)}(0) \) are called the \textit{Euler numbers of order} \( r \), and in the special case, \( r = 1 \), \( E_n^{(1)}(x) = E_n(x) \) are called the \textit{ordinary Euler polynomials}.

By the definition of Euler polynomials of order \( r \), we obtain the following lemma.

\textbf{Lemma 1.1.} For \( n \geq 0 \) and positive integer \( k \), we have
\[
E_n^{(k)}(-x) = (-1)^n E_n^{(k)}(x + k).
\]

\textit{Proof.}
\[
\sum_{n=0}^{\infty} E_n^{(k)}(-x) \frac{t^n}{n!} = \left( \frac{2}{e^t + 1} \right)^k e^{-xt} = \left( \frac{2}{1 + e^{-t}} \right) e^{-(x+k)t} = \sum_{n=0}^{\infty} (-1)^n E_n^{(k)}(x + k) \frac{t^n}{n!}. \tag{1.8}
\]

\( \square \)

We observe that, by (1.2),
\[
\int_{\mathbb{Z}_p} e^{(x+y)t} \ d\mu_{-1}(y) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!},
\]
and thus
\[
\int_{\mathbb{Z}_p} (x + y)^n \ d\mu_{-1}(y) = E_n(x), \quad (n \geq 0), \quad (\text{see [4, 10]}). \tag{1.9}
\]

For \( n \in \mathbb{N} \), let \( T_p \) be the \( p \)-adic locally constant space defined by
\[
T_p = \bigcup_{n \geq 1} C_p^n = \lim_{n \to \infty} C_p^n,
\]
where \( C_p^n = \{ \omega | \omega^{p^n} = 1 \} \) is the cyclic group of order \( p^n \).

We assume that \( q \) is an indeterminate in \( \mathbb{C}_p \) with \( |1 - q|_p < p^{-\frac{1}{n-1}} \). Then we define the \( q \)-analogue of falling factorial sequence as follows:
\[
(x)_{n,q} = x(x - q)(x - 2q) \cdots (x - (n-1)q), \quad (n \geq 1), \quad (x)_{0,q} = 1.
\]
Note that
\[ \lim_{q \to 1} (x)_{n,q} = (x)_n = \sum_{l=0}^{n} S_1(n,l)x^l. \]

Recently, D. S. Kim et. al introduced the Changhee polynomials as follows:

\[ C_n(x) = \int_{\mathbb{Z}_p} (x + y)_n d\mu_1(y), \quad (n \geq 0), \quad (\text{see } [6, 9, 12]). \] (1.10)

When \( x = 0 \), \( C_n = C_n(0) \) are called the \( n \)'s Changhee numbers. From (1.10), we can derive the generating function to be

\[ \left( \frac{2}{2 + t} \right) (1 + t)^x = \sum_{n=0}^{\infty} C_n(x) \frac{t^n}{n!}, \quad (\text{see } [6]). \] (1.11)

In addition, D. S. Kim et. al. consider the Changhee polynomials with \( q \)-parameter which is defined by the generating function to be

\[ \sum_{n=0}^{\infty} C_{n,q} \frac{t^n}{n!} = (1 + qt)^{\frac{x}{q}} \frac{2}{(1 + qt)^{\frac{1}{q}} + 1}, \quad (\text{see } [3]). \] (1.12)

When \( x = 0 \), \( C_{n,q} = C_{n,q}(0) \) are called the Changhee numbers with \( q \)-parameter.

In the viewpoint of generalization of the Changhee polynomials with \( q \)-parameter, we consider the twisted Changhee polynomials with \( q \)-parameter which are defined by the generating function to be

\[ \sum_{n=0}^{\infty} C_{n,\xi,q} \frac{t^n}{n!} = (1 + q\xi t)^{\frac{x}{q}} \frac{2}{(1 + q\xi t)^{\frac{1}{q}} + 1}, \quad \xi \in T_p. \] (1.13)

In this paper, we give a \( p \)-adic integral representation of the twisted Changhee polynomials with \( q \)-parameter, which are called the Witt-type formula for the twisted Changhee polynomials with \( q \)-parameter. We can derive some interesting properties related to the \( n \)-th twisted Changhee polynomials with \( q \)-parameter.

### 2 Witt-type formula for the \( n \)-th twisted Changhee polynomials with \( q \)-parameter

In this section, we assume that \( t \in \mathbb{C}_p \) with \( |t|_p < \left| \frac{p-1}{q} \right|_p \). We consider the following integral representation associated with falling factorial sequences:
\[ \int_{Z_p} (x + y)_{n,q} d\mu_1(y), \text{ where } n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}. \] (2.1)

By (2.1),
\[
\sum_{n=0}^{\infty} \xi^n \int_{Z_p} (x + y)_{n,q} d\mu_1(y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \xi^n q^n \int_{Z_p} \left( \frac{x + y}{q} \right)_n d\mu_1(y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (q\xi)^n \int_{Z_p} \left( \frac{x + y}{q} \right)_n d\mu_1(y) t^n = \int_{Z_p} (1 + q\xi t)^\frac{x + y}{q} d\mu_1(y). \] (2.2)

If we put \( f(x) = (1 + q\xi t)^\frac{x + y}{q} \), then, by (1.1), we get
\[
\int_{Z_p} (1 + q\xi t)^\frac{x + y}{q} d\mu_1(y) = (1 + q\xi t)^\frac{\xi}{q} \frac{2}{(1 + q\xi t)^\frac{\xi}{q} + 1} = \sum_{n=0}^{\infty} (q\xi)^n \int_{Z_p} (1 + q\xi t)^\frac{x + y}{q} d\mu_1(y). \tag{2.3}
\]

By (2.2) and (2.3), we obtain the following theorem.

**Theorem 2.1.** For \( n \geq 0 \), we have
\[ Ch_{n,\xi,q}(x) = \xi^n \int_{Z_p} (x + y)_{n,q} d\mu_1(y). \]

In (2.3), by replacing \( t \) by \( \frac{1}{\xi q} \left( e^{\xi t} - 1 \right) \), we have
\[
\sum_{n=0}^{\infty} Ch_{n,\xi,q}(x) \frac{1}{\xi^n q^n} \frac{(e^{\xi t} - 1)^n}{n!} = \frac{e^{\xi t} - 1}{e^{\xi t} + 1} = \sum_{n=0}^{\infty} E_n(x) \xi^n t^n \frac{q^n}{q^n n!} \tag{2.4}
\]

and
\[
\sum_{n=0}^{\infty} Ch_{n,\xi,q}(x) \frac{1}{\xi^n q^n} \frac{(e^{\xi t} - 1)^n}{n!} = \sum_{n=0}^{\infty} Ch_{n,\xi,q}(x) \frac{1}{\xi^n q^n} \sum_{m=n}^{\infty} \xi^m S_2(m, n) t^m \frac{m!}{m!} = \sum_{m=0}^{\infty} \sum_{n=0}^{m} Ch_{n,\xi,q}(x) \xi^m S_2(m, n) t^m \frac{m!}{m!}. \tag{2.5}
\]

By (2.4) and (2.5), we obtain the following corollary.
Corollary 2.2. For \( n \geq 0 \), we have

\[
E_n(x) = \sum_{m=0}^{n} C_{m,\xi,q}(x) \xi^{-m} q^{n-m} S_2(n, m) \quad (m \geq 0).
\]

By the Theorem 2.1,

\[
C_{n,\xi,q}(x) = \xi^n \int_{\mathbb{Z}_p} (x + y)_{n,q} d\mu_{-1}(y)
\]

\[
= \xi^n q^n \int_{\mathbb{Z}_p} \left( \frac{x + y}{q} \right)_n d\mu_{-1}(y) \quad (2.6)
\]

\[
= \xi^n q^n \sum_{l=0}^{n} \frac{1}{q^l} S_1(n, l) \int_{\mathbb{Z}_p} (x + y)^l d\mu_{-1}(y).
\]

By (1.9) and (2.6), we obtain the following corollary.

Corollary 2.3. For \( n \geq 0 \), we have

\[
C_{n,\xi,q}(x) = \xi^n \sum_{l=0}^{n} q^{n-l} S_1(n, l) E_l(x)
\]

\[
= \xi^n \sum_{l=0}^{n} |S_1(n, l)| (-q)^{n-l} E_l(x).
\]

From now on, we consider twisted Changhee polynomials of order \( k \in \mathbb{N} \) with \( q \)-parameter. Twisted Changhee polynomials of order \( k \in \mathbb{N} \) with \( q \)-parameter are defined by the multivariant \( p \)-adic invariant integral on \( \mathbb{Z}_p \):

\[
C_{n,\xi,q}^{(k)}(x) = \xi^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)_{n,q} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) \quad (2.7)
\]

where \( n \) is an nonnegative integer and \( k \in \mathbb{N} \). In the special case, \( x = 0 \),
\[C_{n,\xi,q}^{(k)} = C_{n,\xi,q}^{(k)}(0) \] are called the Changhee numbers of order \( k \) with \( q \)-parameter.
From (2.7), we can derive the generating function of $D_{n,\xi,q}^{(k)}(x)$ as follows:

$$
\sum_{n=0}^{\infty} Ch_{n,\xi,q}^{(k)}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \xi^n q^n \int \cdots \int (\frac{x_{1}+\cdots+x_k+x}{q})^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) t^n
$$

$$
= \int \cdots \int (1 + q\xi t) \frac{x_{1}+\cdots+x_k+x}{q} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)
$$

(2.8)

$$
= (1 + q\xi t)^\frac{x}{q} \int \cdots \int (1 + q\xi t) \frac{x_{1}+\cdots+x_k+x}{q} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)
$$

$$
= (1 + q\xi t)^\frac{x}{q} \left( \frac{2}{(1 + q\xi t)^\frac{1}{q} + 1} \right)^k.
$$

Note that, by (2.7),

$$
Ch_{n,\xi,q}^{(k)}(x) = \xi^n q^n \sum_{m=0}^{n} \frac{S_1(n,m)}{q^m} \int \cdots \int (x_1 + \cdots + x_k + x)^m d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k).
$$

(2.9)

Since

$$
\int \cdots \int e^{(x_1+\cdots+x_k+x)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)
$$

$$
= \left( \frac{2}{e^t + 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!},
$$

we can derive easily

$$
E_n^{(k)}(x) = \int \cdots \int (x_1 + \cdots + x_k + x)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k).
$$

(2.10)

Thus, by (2.9) and (2.10), we have

$$
Ch_{n,\xi,q}^{(k)}(x) = \xi^n q^n \sum_{m=0}^{n} \frac{S_1(n,m)}{q^m} E_m^{(k)}(x)
$$

$$
= \xi^n \sum_{m=0}^{n} q^{n-m} S_1(n,m) E_m^{(k)}(x)
$$

(2.11)

$$
= \xi^n \sum_{m=0}^{n} |S_1(n,m)| (-q)^{n-m} E_m^{(k)}(x).
$$
In (2.8), by replacing $t$ by $\frac{1}{q}\left(e^{\xi t} - 1\right)$, we get
\[
\sum_{n=0}^{\infty} Ch_n^{(k)}(x) \frac{(e^{\xi t} - 1)^n}{\xi^n q^n n!} = e^{\frac{\xi x}{q}} \left(\frac{2}{e^t + 1}\right)^k
= \sum_{n=0}^{\infty} \frac{\xi^n E_n^{(k)}(x) t^n}{q^n n!},
\]
(2.12)
and
\[
\sum_{n=0}^{\infty} \frac{Ch_n^{(k)}(x)}{\xi^n q^n} \frac{1}{n!} (e^{\xi t} - 1)^n = \sum_{n=0}^{\infty} \frac{Ch_n^{(k)}(x)}{\xi^n q^n} \sum_{l=n}^{\infty} S_2(l, n) t^l l!
= \sum_{m=0}^{\infty} \left(\xi^m \sum_{n=0}^{m} \frac{Ch_n^{(k)}(x)}{\xi^n q^n} S_2(m, n)\right) \frac{t^m}{m!}.
\]
(2.13)
By (2.11), (2.12) and (2.13), we obtain the following theorem.

**Theorem 2.4.** For $n \geq 0$ and $k \in \mathbb{N}$, we have
\[
Ch_n^{(k)}(x) = \xi^n \sum_{m=0}^{n} q^{n-m} S_1(n, m) E_m^{(k)}(x)
= \xi^n \sum_{m=0}^{n} |S_1(n, m)| (-q)^{n-m} E_m^{(k)}(x).
\]
and
\[
E_n^{(k)}(x) = \sum_{m=0}^{n} Ch_m^{(k)}(x) \xi^{-m} q^{n-m} S_2(n, m).
\]

Now, we consider the twisted Changhee polynomials of the second kind with $q$-parameter as follows:
\[
\tilde{Ch}_n^{(k)}(x) = \xi^n \int_{\mathbb{Z}_p} (-y + x)_{n,q} d\mu_{-1}(y), \quad (n \geq 0).
\]
(2.14)
In the special case, $x = 0$, $\tilde{Ch}_n^{(k)}(0) = \tilde{Ch}_n^{(k)}$ are called the twisted Daehee numbers of the second kind with $q$-parameter.

By (2.14), we have
\[
\tilde{Ch}_n^{(k)}(x) = \xi^n q^n \int_{\mathbb{Z}_p} \left(\frac{-y + x}{q}\right)_n d\mu_{-1}(y),
\]
(2.15)
and so we can derive the generating function of $\widehat{Ch}_{n,\xi,q}(x)$ by (1.1) as follows:

$$\sum_{n=0}^{\infty} \frac{\widehat{Ch}_{n,\xi,q}(x) t^n}{n!} = \sum_{n=0}^{\infty} q^n \xi^n \int_{\mathbb{Z}_p} \left( \frac{-y + x}{q} \right)_n d\mu_{-1}(y) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} q^n \xi^n \int_{\mathbb{Z}_p} \left( \frac{-y + x}{q} \right) d\mu_{-1}(y) t^n$$

$$= \int_{\mathbb{Z}_p} (1 + q\xi t)^{-\frac{y+x}{q}} d\mu_{-1}(y)$$

$$= (1 + q\xi t)^{-\frac{x}{q}} \frac{2}{(1 + q\xi t)^\frac{1}{q} + 1}.$$  \hspace{1cm} (2.16)

From (1.3), (1.6) and (2.15), we get

$$\widehat{Ch}_{n,\xi,q}(x) = q^n \xi^n \int_{\mathbb{Z}_p} \left( \frac{-y + x}{q} \right) d\mu_{-1}(y)$$

$$= q^n \xi^n \int_{\mathbb{Z}_p} \sum_{l=0}^{n} S_1(n,l) \left( \frac{-y + x}{q} \right)^l d\mu_{-1}(y)$$

$$= \xi^n \sum_{l=0}^{n} S_1(n,l)(-1)^l \int_{\mathbb{Z}_p} (y - x)^l d\mu_{-1}(y) q^{n-l}$$

$$= \xi^n \sum_{l=0}^{n} S_1(n,l)(-1)^l E_l(-x) q^{n-l}$$

$$= (-\xi)^n \sum_{l=0}^{n} |S_1(n,l)| E_l(-x) q^{n-l}.$$  \hspace{1cm} (2.17)

By Lemma 1.1 and (2.17), we have the following theorem.

**Theorem 2.5.** For $n \geq 0$, we have

$$\widehat{Ch}_{n,\xi,q}(x) = \xi^n \sum_{l=0}^{n} S_1(n,l)(-1)^l E_l(-x) q^{n-l}$$

$$= \xi^n \sum_{l=0}^{n} |S_1(n,l)| E_l(x+1) (-q)^{n-l}.$$  \hspace{1cm} (2.18)

By replacing $t$ by $\frac{1}{q^\xi} (e^{\xi t} - 1)$ in (2.16), we have

$$\sum_{n=0}^{\infty} \frac{\widehat{Ch}_{n,\xi,q}(x)}{q^n \xi^n} \frac{(e^{\xi t} - 1)^n}{n!} = e^{\xi (x+1) \frac{2}{e^{\frac{\xi}{q}} + 1}}$$

$$= \sum_{n=0}^{\infty} \frac{\xi^n E_n(x+1) t^n}{q^n n!}.$$  \hspace{1cm} (2.18)
and

\[ \sum_{n=0}^{\infty} \hat{C}h_{n,\xi,q}(x) \frac{1}{q^n \xi^n} \frac{(e^{\xi t} - 1)^n}{n!} = \sum_{n=0}^{\infty} \hat{C}h_{n,\xi,q}(x) \sum_{m=n}^{\infty} S_2(m, n) \frac{(\xi t)^m}{m!} \]

\[ = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \hat{C}h_{m,\xi,q}(x) S_2(n, m) q^{-m} \xi^{n-m} \right) \frac{t^n}{n!}. \tag{2.19} \]

By (2.18) and (2.19), we obtain the following theorem.

**Theorem 2.6.** For \( n \geq 0 \), we have

\[ E_n(x + 1) = \sum_{m=0}^{n} q^{n-m} \xi^{-m} \hat{C}h_{m,\xi,q}(x) S_2(n, m). \]

Now, we consider higher-order twisted Changhee polynomials of second kind with \( q \)-parameter. Higher-order twisted Changhee polynomials of second kind with \( q \)-parameter are defined by the multivariant \( p \)-adic invariant integral on \( \mathbb{Z}_p \):

\[ \hat{C}h^{(k)}_{n,\xi,q}(x) = \xi^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - \cdots - x_k + x)_{n,q} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) \tag{2.20} \]

where \( n \) is an nonnegative integer and \( k \in \mathbb{N} \). In the special case, \( x = 0 \), \( \hat{C}h^{(k)}_{n,\xi,q} = \hat{C}h^{(k)}_{n,\xi,q}(0) \) are called the higher-order twisted Changhee numbers of second kind with \( q \)-parameter.

From (2.20), we can derive the generating function of \( \hat{C}h^{(k)}_{n,\xi,q}(x) \) as follows:

\[ \sum_{n=0}^{\infty} \hat{C}h^{(k)}_{n,\xi,q}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \xi^n q^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( \frac{x_1 - \cdots - x_k + x}{q} \right) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) t^n \]

\[ = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + q\xi t)^{-\frac{x_1 - \cdots - x_k + x}{q}} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) \]

\[ = (1 + q\xi t)^{\frac{x+k}{q}} \left( \frac{2}{(1 + q\xi t)^{\frac{1}{q}} + 1} \right)^k. \tag{2.21} \]
By (2.20),
\[
\hat{Ch}_{n,k,q}(x) = \sum_{m=0}^{n} \frac{S_1(n,m)}{q^m} \int_{z_p} \cdots \int_{z_p} (-x_1 - \cdots - x_k + x)^m d\mu_1(x_1) \cdots d\mu_1(x_k)
\]
\[
= \sum_{m=0}^{n} \frac{S_1(n,m)}{(-q)^m} \int_{z_p} \cdots \int_{z_p} (x_1 + \cdots + x_k - x)^m d\mu_1(x_1) \cdots d\mu_1(x_k)
\]
\[
= \xi^n \sum_{m=0}^{n} \frac{S_1(n,m)}{(-q)^m} E_n^{(k)}(-x)
\]
\[
= \xi^n \sum_{m=0}^{n} q^{n-m} S_1(n,m) |E_n^{(k)}(-x)|.
\]

(2.22)

From Lemma 1.1 and (2.22), we obtain the following theorem.

**Theorem 2.7.** For \( n \geq 0 \), we have

\[
\hat{Ch}_{n,k,q}(x) = \xi^n \sum_{m=0}^{n} (-1)^m q^{n-m} S_1(n,m) E_n^{(k)}(-x)
\]
\[
= \xi^n \sum_{m=0}^{n} (-1)^m q^{n-m} |S_1(n,m)| E_n(x + k).
\]

In (2.21), by replacing \( t \) by \( \frac{1}{q\xi}(e^\xi t - 1) \), we get

\[
\sum_{n=0}^{\infty} \hat{Ch}_{n,k,q}(x) \left( \frac{e^\xi t - 1}{q^n} \right) = e^{\xi t(x+1)} \left( \frac{2}{e^\xi + 1} \right)^k
\]
\[
= \sum_{n=0}^{\infty} \frac{\xi^n E_n^{(k)}(x+1) t^n}{q^n n!},
\]

(2.23)

and

\[
\sum_{n=0}^{\infty} \hat{Ch}_{n,k,q}(x) \frac{1}{n!} (e^\xi t - 1)^n = \sum_{n=0}^{\infty} \frac{\hat{Ch}_{n,k,q}(x)}{\xi^n q^n} \sum_{l=n}^{\infty} S_2(l, n) \xi^l t^l n!
\]
\[
= \sum_{m=0}^{\infty} \left( \xi^m \sum_{n=0}^{m} \frac{\hat{Ch}_{n,k,q}(x)}{\xi^n q^n} S_2(m, n) \right) \frac{t^m}{m!}.
\]

(2.24)

By (2.23) and (2.24), we obtain the following theorem.
Theorem 2.8. For $n \geq 0$ and $k \in \mathbb{N}$, we have

$$E_n^{(k)}(x+1) = \sum_{m=0}^{n} \hat{C} Ch^{(k)}_{m,\xi,q}(x)\xi^{-m}q^{n-m}S_2(n,m).$$

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References


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