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Rainbow Connection Number of the Thorn Graph

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Abstract

A path in an edge colored graph is said to be a rainbow path if every edge in this path is colored with the same color. The rainbow connection number of G, denoted by $rc(G)$, is the smallest number of colors needed to color its edges, so that every pair of its vertices is connected by at least one rainbow path. A rainbow $u-v$ geodesic in G is a rainbow path of length $d(u,v)$, where $d(u,v)$ is the distance between u and v. The graph G is strongly rainbow connected if there exists a rainbow $u-v$ geodesic for any two vertices u and v in G. The strong rainbow connection number $src(G)$ of G is the minimum number of colors needed to make G strongly rainbow connected.
In this paper, we determine the exact values of $rc(G)$ and $src(G)$ where G are the thorn graph of complete graph K_n^*, the thorn graph of the cycle C_n^*.

Mathematics Subject Classification: 05C15, 05C40

Keywords: (strong) rainbow connection number, thorn graph, complete graph, cycle

1 Introduction

All graphs considered in this paper are undirected, finite, and simple. Connectivity is perhaps the most fundamental graph-theoretic property. A natural and interesting quantifiable way to strengthen the connectivity requirement was was recently introduced by Chartrand et al. in [1].
An edge coloring of a graph is a function from its edges set to the set of natural numbers. The rainbow connection number of G, denoted by $rc(G)$, is the smallest number of colors needed to color its edges, so that every pair of its vertices is connected by at least one path in which no two edges are colored the same. In this case, the coloring c is called a rainbow coloring of G. For two vertices u and v of G, a rainbow $u-v$ geodesic in G is a rainbow $u-v$ path of length $d(u,v)$, where $d(u,v)$ is the distance between u and v. The graph G is strongly rainbow-connected if G contains a rainbow $u-v$ geodesic for every two vertices u and v of G. In this case, the coloring c is called a strong rainbow coloring of the edges of G. The minimum k for which there exists a coloring c of the edges of G such that G is strongly rainbow-connected is the strong rainbow connection number $src(G)$ of G. Thus, $rc(G) \leq src(G)$ for every connected graph G. Furthermore, if G is a nontrivial connected graph of size m whose diameter denoted by $diam(G)$, then

$$diam(G) \leq rc(G) \leq src(G) \leq m.$$

The rainbow connection of fan, sun and some corona graphs have been studied in [2, 3]. There have been some results on the (strong) rainbow connection numbers of graphs (see [1, 4]).
In this paper, we will show that $rc(K_n^*) = src(K_n^*) = \sum_{i=1}^{n} l_i$, where K_n^* is the thorn graph of the complete graph. Moreover, if G is the thorn graph of the cycle, then $rc(G) = src(G) = \lfloor \frac{n}{2} \rfloor + \sum_{i=1}^{n} l_i$ $(n \geq 4)$.

2 Preliminary Notes

At the beginning of this paper, we first introduce the definition of the thorn of the graph.

Definition 2.1. *[5] Let $l_1, l_2, \ldots, l_n$ be positive integers and G be such a graph, $V(G) = \{v_1, v_2, \ldots, v_n\}$. The thorn of the graph G, with parameters $l_1, l_2, \ldots, l_n$, is obtained by attaching l_i new vertices of degree 1 to the vertex v_i of the graph G $(i \in \{1, \ldots, n\})$.*

The thorn graph of the graph G will be denoted by G^* or by $G^*(l_1, l_2, \ldots, l_n)$, if the respective parameters need to be specified.

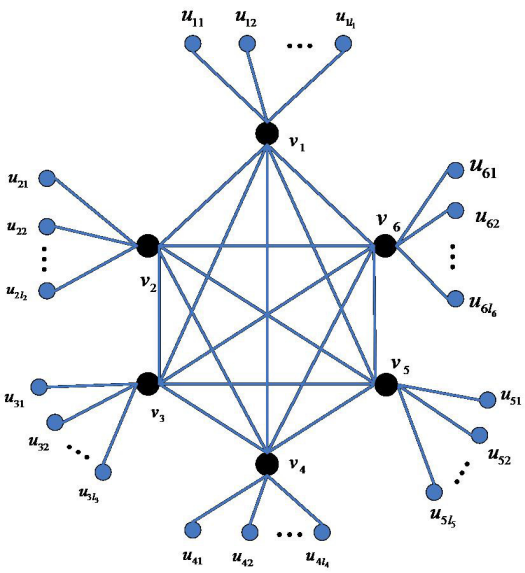

Figure 1: The thorn graph of K_6

3 Main Results

In this paper, we will consider the thorn graph with every $l_i \geq n$ ($i \in \{1, \ldots, n\}$).

Theorem 3.1. *For a positive integer n we have that*

$$rc(K_n^*) = src(K_n^*) = \sum_{i=1}^{n} l_i.$$

Proof. Consider u_{ij} are the thorn which come from the vertex v_i ($i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, l_i\}$). See Fig. 1.
First, we show that $rc(K_n^*) \leq \sum_{i=1}^{n} l_i$. Since all paths from $u_{i_1 j_1}$ to $u_{i_2 j_2}$ need to go through the edges $u_{i_1 j_1} v_{i_1}$, $v_{i_2} u_{i_2 j_2}$, it is obvious that the color of the edges $v_i u_{ij}$ must be different($i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, l_i\}$). In other words, this is a necessary condition for a graph's rainbow connectivity. Consequently, in this coloring, we color all the thorn edges $v_i u_{ij}$ as $c(v_i u_{ij}) = j^{(i)}$ firstly, where $j^{(i)}$ $i \in \{1, \ldots, n\}, j \in \{1, \ldots, l_i\}$ is the color codes of the edges of the graph. We can color the other edges that we hade left as follows:

$$\begin{cases} c(v_i v_j) = (j+1)^{(i+1)}, & \text{for } i < j < n \ i \in \{1, \ldots, n\}, j \in \{1, \ldots, l_i\}, \\ c(v_i v_n) = 1^{(i+1)}, & \text{for } i \in \{1, \ldots, n\}, \\ c(v_n v_1) = 2^{(2)}. \end{cases}$$

It is obvious that the paths of $u_{i_1 j_1} v_{i_1} - v_{i_1} v_{i_2} - v_{i_2} u_{i_2 j_2}$ are colored with $j_1^{(i_1)} - (i_2+1)^{(i_1+1)} - j_2^{(i_2)}$, those $u_{i_1 j_1} v_{i_1} - v_{i_1} v_n - v_n u_{n j_2}$ are colored with the colors

of $j_1^{(i_1)} - 1^{(i_1+1)} - j_2^{(n)}$. $j_1^{(n)} - 2^{(2)} - j_2^{(1)}$ are the color codes of the paths $u_{nj_1}v_n - v_nv_1 - v_1u_{1j_2}$. All the paths of the graph are included in the paths we have discusses above. In other words, any two vertices of the graph K_n^* are connected by a rainbow path with the $\sum_{i=1}^n l_i$ colors. So, we can construct a rainbow $\sum_{i=1}^n l_i$-coloring of K_n^* that makes the graph K_n^* rainbow connected. This implies that $rc(K_n^*) \leq \sum_{i=1}^n l_i$.
Next, we show that $rc(K_n^*) \geq \sum_{i=1}^n l_i$. Assume, to the contrary, that $rc(K_n^*) \leq \sum_{i=1}^n l_i - 1$. Let c^1 be a rainbow $(\sum_{i=1}^n l_i - 1)$-coloring of K_n^*. We suppose that $u_{i_1j_1}v_{i_1}$ and $v_{i_2}u_{i_2j_2}$ are two edges with the same color, then the path $u_{i_1j_1} - u_{i_2j_2}$ in K_n^* is not a rainbow path. This leads to a contradiction. Therefore, $rc(K_n^*) = \sum_{i=1}^n l_i$.
Second, we show that the rainbow coloring c is a strong rainbow coloring.
Since $u_{i_1j_1} - v_{i_1}v_{i_2} - u_{i_2j_2}$ is the only path with length $d(u_{i_1j_1}, u_{i_2j_2})$ between $u_{i_1j_1}$ and $u_{i_2j_2}$ in K_n^*, where $i_1, i_2 \in \{1, \ldots, n\}$ and $i_1 \neq i_2$, $j_1 \in \{1, \ldots, l_{i_1}\}$, $j_2 \in \{1, \ldots, l_{i_2}\}$, it follows that the rainbow coloring c can make K_n^* strongly rainbow connected. In other words, $src(K_n^*) \leq \sum_{i=1}^n l_i$.
To show that $src(K_n^*) > \sum_{i=1}^n l_i$, we suppose, without loss of generality, that $src(K_n^*) = \sum_{i=1}^n l_i - 1$. There are at least two thorn edges colored the same, and then the rainbow geodesic from one thorn to another does not exist, which is a contradiction.
The proof is thus complete. □

Theorem 3.2. *For integer $n \geq 4$, we have that*

$$rc(C_n^*) = src(C_n^*) = \lfloor \frac{n}{2} \rfloor + \sum_{i=1}^n l_i.$$

Proof. For $n \geq 4$, we consider the cycle C_n with n vertices and for each i with $1 \leq i \leq n$, let $e_i = v_iv_{i+1}$. We define an edge-coloring c of C_n as follows:

$$c(e_i) = \begin{cases} i, & \text{if } 1 \leq i \leq \lceil n/2 \rceil, \\ i - \lceil n/2 \rceil, & \text{if } \lceil n/2 \rceil + 1 \leq i \leq n. \end{cases}$$

Since the color of the thorn edges must be different, we have that $rc(C_n^*) \leq \lfloor \frac{n}{2} \rfloor + \sum_{i=1}^n l_i$.
Next, to show that $rc(C_n^*) \geq \lfloor \frac{n}{2} \rfloor + \sum_{i=1}^n l_i$, we assume to contrary that $rc(C_n^*) \leq \lfloor \frac{n}{2} \rfloor + \sum_{i=1}^n l_i - 1$. Let c^* be a rainbow $(\lfloor \frac{n}{2} \rfloor + \sum_{i=1}^n l_i - 1)$-coloring of C_n^*. Without loss of generality, assume that $c^*(e_i) = c(e_i)$ then there are at least two thorn edges colored the same. On the other hand, if $c^*(v_iu_{ij}) = c(v_iu_{ij})$ then the $(\lfloor \frac{n}{2} \rfloor + \sum_{i=1}^n l_i - 1)$-coloring of C_n^* that assigns the $\lfloor \frac{n}{2} \rfloor - 1$ distinct colors to the remaining n edges of C_n is not a rainbow coloring. Therefore, as claimed, $rc(C_n^*) = \lfloor \frac{n}{2} \rfloor + \sum_{i=1}^n l_i$.
Next, we show that $src(C_n^*) = \lfloor \frac{n}{2} \rfloor + \sum_{i=1}^n l_i$. Since $diam(C_n) = \lfloor \frac{n}{2} \rfloor$ and the color of the thorn edges must be different, we can conclude that $src(C_n^*) \leq$

$\lfloor \frac{n}{2} \rfloor + \sum_{i=1}^{n} l_i$. Next, we show that $src(C_n^*) \geq \lfloor \frac{n}{2} \rfloor + \sum_{i=1}^{n} l_i$. We assume to contrary that $src(C_n^*) \leq \lfloor \frac{n}{2} \rfloor + \sum_{i=1}^{n} l_i - 1$. Let c^1 be a rainbow $(\lfloor \frac{n}{2} \rfloor + \sum_{i=1}^{n} l_i - 1)$-coloring of C_n^*. Without loss of generality, assume that $c^1(e_i) = c(e_i)$ then there are at least two thorn edges colored the same as well. Generally, assume that $c^1(u_{ij}) = c(u_{mn})(i \neq m \in \{1, 2, \cdots, n\})$, then there is no $u_{ij}v_i - v_m u_{mn}$ geodesic in C_n^*. Similar to the proof of the rainbow connection number, assume that $c^1(v_i u_{ij}) = c(v_i u_{ij})$ then this is in contradiction to $src(C_n) = \lfloor \frac{n}{2} \rfloor$. By this possibilities have been exhausted and the proof is complete. □

4 Conclusion

In Theorem 3.1 and Theorem 3.2, we have calculated the (strong) rainbow connection number of the thorn graph of the complete graph and the cycle respectively. What happens with the thorn graph of any special graphs? Future research may be done on the rainbow connection number of the thorn graph of any special graphs. That is another problem: what kind of graphs G meet that $rc(G^*) = rc(G) + \sum l_i$, where $\sum l_i$ is the total number of the thorn. Finally, the condition of the equation $rc(G) = src(G)$ need further discussion.

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