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The Rainbow Connection of Windmill and Corona Graph

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Abstract

The rainbow connection number of G, denoted by $rc(G)$, is the smallest number of colors needed to color its edges, so that every pair of its vertices is connected by at least one path in which no two edges are colored the same. A rainbow $u-v$ geodesic in G is a rainbow path of length $d(u,v)$, where $d(u,v)$ is the distance between u and v. The graph G is strongly rainbow connected if there exists a rainbow $u-v$ geodesic for any two vertices u and v in G. The strong rainbow connection number $src(G)$ of G is the minimum number of colors needed to make G strongly rainbow connected.
In this paper we determine the exact values of the windmill graph $K_m^{(n)}$. Moreover, we compute the $rc(G \circ H)$ where G or H is complete graph K_m or path P_2 with m is an integer. These graphs we studied have progressive connection on structure.

Mathematics Subject Classification: 05C15, 05C40

Keywords: (strong) rainbow connection number, windmill graph, corona graph

1 Introduction

An edge coloring of a graph is a function from its edges set to the set of natural numbers. The concept of rainbow connectivity was recently introduced by Chartrand et al. in [1]. A path is rainbow if no two edges of it are colored the same. An edge-coloring graph G is rainbow connected if any two vertices are connected by a rainbow path. The minimum k for which there exists a rainbow k-coloring of the edges of G is the rainbow connection number $rc(G)$ of G. For two vertices u and v of G, a rainbow $u-v$ geodesic in G is a rainbow $u-v$ path of length $d(u,v)$, where $d(u,v)$ is the distance between u and v. The graph G is strongly rainbow-connected if G contains a rainbow $u-v$ geodesic for every two vertices u and v of G. The minimum k for which there exists a coloring c of the edges of G such that G is strongly rainbow-connected is the strong rainbow connection number $src(G)$ of G. Thus, $rc(G) \leq src(G)$ for every connected graph G. Furthermore, if G is a nontrivial connected graph of size m whose diameter denoted by $diam(G)$, then $diam(G) \leq rc(G) \leq src(G) \leq m$. The rainbow connection of fan, sun and some corona graphs have been studied in [2, 3]. There have been some results on the (strong) rainbow connection numbers of graphs (see [1, 4]). In this paper, we will give the (strong) rainbow connection numbers of the windmill graph and corona graph.

2 Preliminary Notes

At the beginning of this paper, we first introduce some preliminary notes used in the paper.

Definition 2.1. *[5] The windmill graph $K_m^{(n)}$ is the graph consisting of n copies of the complete graph K_m with a vertex in common.*

Definition 2.2. *[3] The corona $G \circ H$ of two graphs G and H (where $|V(G)| = m$ and $|V(H)| = n$) is defined as the graph G obtained by taking one copy of G and copies of H, called $H_1, H_2, \ldots, H_m$, and then joining by a line the i'th vertex of G to every vertex in the i'th copy of H.*

3 Results and Discussion

At the beginning of this part, we will find the rainbow connection number and the strong rainbow connection number of windmill graph.

Theorem 3.1. *For positive integers m and n we have that*

$$rc(K_m^{(n)}) = \begin{cases} 1, & n = 1, \\ 2, & n = 2, \\ 3, & n \geq 3. \end{cases}$$

the strong rainbow connection number of the windmill graph is $src(K_m^{(n)}) = n$.

Proof. Let v_0 be the center vertex and u_{ij} be the vertices of the i'th complete graph($i \in \{1, \ldots, n\}$ $j \in \{1, \ldots, m-1\}$).

First, we show the proof of the rainbow connection number. Since the proofs of $n = 1, 2$ are obvious, we consider the situation where $n \geq 3$. Because of $diam(K_m^{(n)}) = 2$, $rc(K_m^{(n)}) \geq diam(K_m^{(n)}) = 2$. Assume $rc(K_m^{(n)}) = 2$, then there exists a rainbow 2-coloring $c: E(K_m^{(n)}) \to \{1, 2\}$. Without loss of generality, we let $c(v_0u_{11}) = 1$. $u_{11} - v_0 - u_{21}$ is the only $u_{11} - u_{21}$ path of length 2 in $K_m^{(n)}$. There is a rainbow path from u_{11} to u_{21} if and only if $c(v_0u_{21}) = 2$. However, there is no rainbow path from u_{21} to u_{31} in this case. Hence, $rc(K_m^{(n)}) \geq 3$. We distinguish two cases to show that $rc(K_m^{(n)}) = 3$.

Before giving proof, we should explain a symbol. First, each complete graph is labeled according to clockwise. Then, K_m^i is the symbol of the i'th complete graph.

Case 1 $n = 3$, to show that $rc(K_m^{(3)}) \leq 3$, we provide a rainbow 3-coloring $c: E(K_m^{(n)}) \to \{0, 1, 2\}$ defined by $c(K_m^i) = i - 1$ $(i \in \{1, 2, 3\})$. This implies that all the edges of the i'th complete graph are colored with the color $i - 1$. Therefore, $rc(K_m^{(3)}) = 3$.

Case 2 $n > 3$, we define a 3-coloring $c: E(K_m^{(n)}) \to \{0, 1, 2\}$, let any three of the complete graphs be colored 0, 1, 2 respectively. Without loss of generality, let $c(K_m^k) = k - 1$ $(k \in \{1, 2, 3\})$. Now, we color the rest of the complete graphs as follows:

$$\begin{cases} c(v_0u_{ij}) = j \bmod 3, \\ c(u_{ij}u_{ij+1}) = \{0, 1, 2\} - c(v_0u_{ij}) - c(v_0u_{ij+1}), \\ c(e) \in \{0, 1, 2\}, \text{ for } e \in E(G) - \{v_0u_{ij}\} - \{u_{ij}u_{ij+1}\}. \end{cases}$$

Where $i \in \{4, 5, \ldots, n\}$, $j \in \{1, 2, \ldots, m-1\}$. In this coloring, all the edges of a complete graph are colored with at least one color. So, each complete graph is rainbow connected under the coloring we have defined above. Meanwhile, each path from one vertex $u_{i_1j_1}$ of a complete graph to a vertex $u_{i_2j_2}$ of another complete graph is colored according to the following three conditions. See Fig. 1.

This implies that $rc(K_m^{(n)}) \leq 3$. Thus $rc(K_m^{(n)}) = 3$ $(n \geq 3)$.

Second, we show that $src(K_m^{(n)}) = n$. Since $u_{11} - v_0 - u_{21}$ is the only $u_{11} - u_{21}$ path with length $d(u_{11}u_{21})$ between u_{11} and u_{21} in $K_m^{(n)}$, what if $u_{11} - v_0 - u_{21}$ is a geodesic, the edges $v_0 - u_{11}$, $v_0 - u_{21}$ need to be colored different. In the procedure, the edges $v_0 - u_{i1}$ $(i \in \{1, \ldots, n\})$ have distinct colors. $src(K_m^{(n)}) \geq n$ was established. Now we provide a strong rainbow n-coloring $c^*: E(K_m^{(n)}) \to \{1, 2, \cdots, n\}$ of $K_m^{(n)}$ defined by $c^*(K_m^{(i)}) = i$ $(i \in \{1, \ldots, n\})$. This suggests $src(K_m^{(n)}) \leq n$. Hence the strong rainbow connection number is $src(K_m^{(n)}) = n$. $\square$

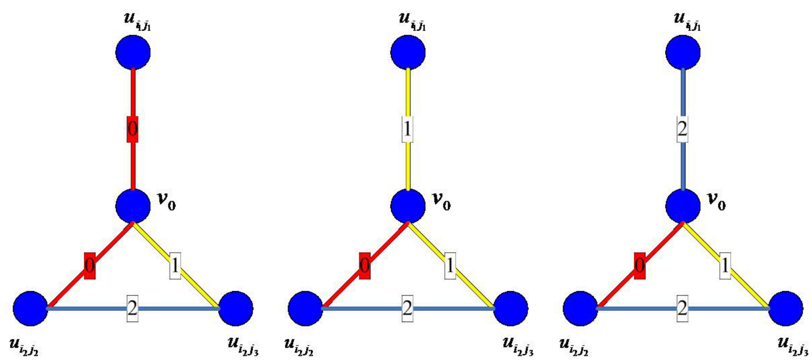

Figure 1: The three cases of the rainbow path

When the sharing vertex of the windmill graph replaced by the complete graph, we can derive some special corona graphs. In the following, we determine the rainbow connection numbers of some corona graphs for combination of some complete graphs and P_2.

Theorem 3.2. *The rainbow connection number of $G \circ H$ is*

$$rc(G \circ H) = \begin{cases} \begin{array}{ll} 3, \quad m = 3, \\ 4, \quad m \geq 4. \end{array} & \textit{for } (G \cong K_m \textit{ and } H \cong P_2), \\ \begin{array}{ll} 3, \quad m = 3, \\ 4, \quad m \geq 4. \end{array} & \textit{for } (G \cong K_m \textit{ and } H \cong K_n \textit{ with } n \geq 3). \end{cases}$$

Proof. We consider two cases.

Case 1 For $G \cong K_m$ and $H \cong P_2$. Since $diam(K_m \circ P_2) = 3$, then $rc(K_m \circ P_2) \geq diam(K_m \circ P_2) = 3$. For $m = 3$, we define a rainbow 3-coloring $c: \ E(K_3 \circ P_2) \to \{1, 2, 3\}$ of $K_m \circ P_2$ as follows:

$$\begin{cases} c(v_m u_{m1}) = c(v_m u_{m2}) = c(u_{m1} u_{m2}) = m, & m \in \{1, 2, 3\}, \\ c(v_i v_j) = \{1, 2, 3\} - \{i, j\}, & i \neq j \text{ and } i, j \in \{1, 2, 3\}. \end{cases}$$

This implies that $rc(K_3 \circ P_2) \leq 3$. Thus $rc(K_3 \circ P_2) = 3$. For $m \geq 4$, if $rc(K_m \circ P_2) = 3$, there will always exists a non-rainbow path $u_{is} v_i - v_i - v_j - v_j u_{jt}$ that the edges $u_{is} v_i$ and $v_j u_{jt}$ are colored with the same color. Thus $rc(K_m \circ P_2) > 3$. We can construct a rainbow 4-coloring $c: \ E(K_m \circ P_2) \to \{1, 2, 3, 4\}$ defined by:

$$\begin{cases} c(v_m u_{m1}) = c(v_m u_{m2}) = c(u_{m1} u_{m2}) = m, & \text{for } m \in \{1, 2, 3, 4\}, \\ \quad c(v_m u_{m1}) = 1, & \\ \quad c(v_m u_{m2}) = 3, & \text{for } m \in \{5, 6, \ldots, n\}, \\ \quad c(u_{m1} u_{m2}) = 2. & \\ c(v_i v_j) \in \{1, 2, 3, 4\} - \{i, j\}, & \text{for } i \neq j \ i, j \in \{1, 2, 3, 4\}, \\ c(v_p v_q) = 4, & \text{for } 4 \neq p \in \{1, \ldots, n\} \text{ and } q \in \{5, 6, \ldots, n\}, \\ c(v_4 v_q) \in \{1, 2, 3\}, & \text{for } q \in \{5, 6, \ldots, n\}. \end{cases}$$

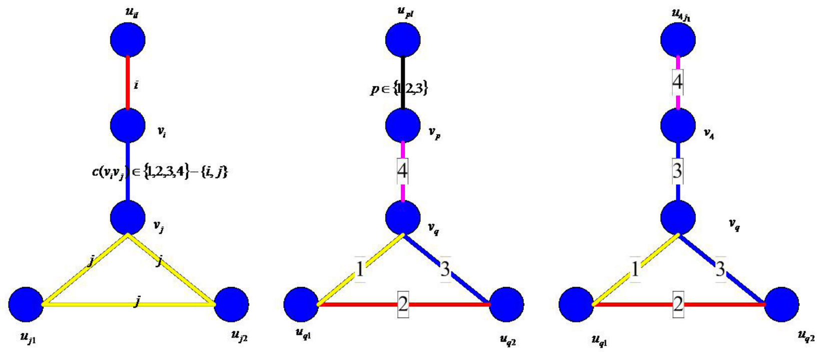

Figure 2: The case of all edges of $E(K_m \circ P_2)$ $m \geq 4$ are different colored

Similar to the theorem 3.1. All edges of $E(K_m \circ P_2)$ $m \geq 4$ are different colored as follow. See Fig. 2. It implies that $rc(K_m \circ P_2) \leq 4$. Thus, $rc(K_m \circ P_2) = 4$.
Case 2 For $G \cong K_m$ and $H \cong K_n$.
Since $diam(K_m \circ K_n) = 3$, then $rc(K_m \circ K_n) \geq diam(K_m \circ K_n) = 3$. For $m = 3$, to show $rc(K_3 \circ K_n) \leq 3$, we define a rainbow 3-coloring $c : \ E(K_3 \circ K_n) \to \{1, 2, 3\}$ of $K_m \circ K_n$:

$$\begin{cases} c(v_m u_{ml}) = c(u_{ml} u_{mk}) = m, & m \in \{1,2,3\} \;\; l \neq k \text{ and } l, k \in \{1, 2, \ldots, n-1\}, \\ c(v_i v_j) \in \{1,2,3\} - \{i, j\}, & i \neq j \;\; i, j \in \{1, 2, 3\}. \end{cases}$$

Thus $rc(K_3 \circ K_n) = 3$. For $m \geq 4$, similar to Case 1, $rc(K_3 \circ K_n) > 3$. To verify $rc(K_3 \circ K_n) \leq 4$, we provide a rainbow 4-coloring $c : \ E(K_m \circ K_n) \to \{1, 2, 3, 4\}$ of $K_m \circ K_n$ defined by

$$\begin{cases} c(v_m u_{ml}) = c(u_{ml} u_{mk}) = m, & \text{for } m \in \{1,2,3,4\} \;\; l \neq k \text{ and } l, k \in \{1, \ldots, n-1\}, \\ \quad c(v_m u_{ml}) = 2, & \\ \quad c(u_{m1} u_{m2}) = 3, & \text{for } m \in \{5, 6, \ldots, n\} \text{ and } l \in \{1, 2, \ldots, n-1\}, \\ \quad c(u_{m1} u_{m3}) = 4, & \\ \quad c(u_{m2} u_{m3}) = 1. & \\ c(v_i v_j) \in \{1,2,3,4\} - \{i, j\}, & \text{for } i \neq j \;\; i, j \in \{1, 2, 3, 4\}, \\ \quad c(v_p v_q) = 4, & \text{for } p \in \{1,2,3\} \text{ and } q \in \{5, 6, \ldots, n\}, \\ \quad c(v_4 v_q) = 2. & \\ c(v_s v_t) = 4, & \text{for } s \neq t \text{ and } s, t \in \{5, 6, \ldots, n\}. \end{cases}$$

Therefore, $rc(K_m \circ K_n) = 4$ for $m \geq 4$ as well. □

4 Conclusion

In this paper, the (strong) rainbow connection number of the windmill graph is calculated. Moreover, we determine the exact values of $rc(G \circ H)$ where

G and H are complete graph K_m and path P_2 with m is an integer. Future research may be done on the strong rainbow connection numbers of the corona graphs.

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