On the Numerical Solution of Coefficient Identification Problem in Heat Equation

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Abstract

In this paper, we investigate the time dependent coefficient identification in heat equation subject to over-specification data at a point in the spatial domain \( u(x^*, t) = G(t), \ 0 \leq t \leq T \) along with usual initial and boundary conditions. Finite difference method is used to determine the solution and time dependent coefficient. A number of numerical illustrations are given to justify the proposed computational scheme.

Mathematics Subject Classification: 35R25; 35R30

Keywords: Identification, Over-specification, Heat equation, Finite-difference scheme


1 Introduction

In recent years, coefficient and source identification problems in parabolic equations have received considerable attention in several fields such as quantum mechanics, finance, thermoelasticity, chemical diffusion, fluid dynamics, and control theory. Some detailed treatments of problems in these fields are carried out in [4], [5], [9], [10], and [11].

Let $\Omega = (0, L)$ be an open bounded set of real numbers. Let us consider the following parabolic partial differential equation with initial and boundary conditions

\[
\frac{\partial u(x, t)}{\partial t} = a(t) \frac{\partial^2 u(x, t)}{\partial x^2}, \quad (x, t) \in Q
\]

\[
u(0, t) = \psi_1(t), \quad u(L, t) = \psi_2(t), \quad t \in (0, T)
\]

\[
u(x, 0) = u_0(x), \quad x \in \Omega
\]

\[
u(x^*, t) = G(t), \quad 0 < t \leq T, \quad 0 \leq x^* \leq L
\]

where $u$ is a function of $x$ and $t$, $T > 0$, $Q = (0, T) \times \Omega$, $\psi_1(t)$, $\psi_2(t)$, and $u_0(x)$ are known $L^2$ functions. Similarly $G(t) \neq 0 \in C^1(0, T)$, the over specification at point $x^*$ in the spatial domain, is also known. Several methods, [2], [12], and [13], have been developed for the source identification problem. Some methods have also been developed for coefficient identification. For example, in [14], the thermal diffusion coefficient $a(t)$ was identified using the shifted Legendre-Tau method. In [6] and in [7] a control parameter is estimated using different numerical techniques which are then compared; both examined a less common difference method achieving accurate results. Coefficient identification in nonlinear equations has been developed as well. In [3], the diffusion coefficient in a nonlinear diffusion equation is identified with respect to overspecified data. In this paper, we study an inverse problem to determine $\{a(t), u(x, t)\}$ in (1.1) – (1.4) from overspecification $G(t)$ along with initial and boundary data $\psi_1(t)$, $\psi_2(t)$, $u_0(x)$.

This paper is organized as follows. In section 2, we introduce inverse problem with overspecification data. Finite difference method with computational algorithm is presented in section 3, and numerical results are presented in section 4. Conclusions of this work are presented in section 5.
2 The Inverse Problem with Overspecification Data

Since $G(t) \in C^1(0, \infty)$, we differentiate overspecification function in (1.4) with respect to $t$. We get,

$$\frac{\partial u(x^*, t)}{\partial t} = \frac{dG(t)}{dt}$$  \hspace{1cm} (5)

It is assumed that (1.1) holds for the spatial domain $\Omega$. Thus for $x^* \in \Omega$ we have,

$$\frac{\partial u(x^*, t)}{\partial t} = a(t) \frac{\partial^2 u(x^*, t)}{\partial x^2}, \quad (x, t) \in Q$$  \hspace{1cm} (6)

From equations (2.1) and (2.2) we have,

$$\frac{dG(t)}{dt} = a(t) \frac{\partial^2 u(x^*, t)}{\partial x^2}, \quad (x^*, t) \in Q$$  \hspace{1cm} (7)

Thus the unknown coefficient $a(t)$ can be expressed in terms of overspecification function as mentioned in (1.4).

$$a(t) = \frac{dG(t)}{dt} \bigg/ \frac{\partial^2 u(x^*, t)}{\partial x^2}, \quad (x^*, t) \in Q$$  \hspace{1cm} (8)

From equations (1.1) and (2.4) we have the following initial boundary value problem.

$$\frac{\partial u(x, t)}{\partial t} = \left( \frac{dG(t)}{dt} \bigg/ \frac{\partial^2 u(x^*, t)}{\partial x^2} \right) \frac{\partial^2 u(x, t)}{\partial x^2}, \quad (x, t) \in Q$$  \hspace{1cm} (9)

$$u(0, t) = \psi_1(t), \quad u(L, t) = \psi_2(t), \quad t \in (0, T)$$  \hspace{1cm} (10)

$$u(x, 0) = u_0(x), \quad x \in \Omega$$  \hspace{1cm} (11)

To approximate unknown coefficient $a(t)$, we solve equations (2.5) – (2.7) by finite difference method described as in section 3.

3 Finite Difference Method

The finite difference method obtains an approximate solution $(u(x_i, t_j))$ for $u(x, t)$ of (2.5) – (2.7) at a finite set of $x$ and $t$. Thus, we partition the spatial domain $[0, L]$ into $M$ equal subintervals of length $\Delta x$ resulting in a set...
of discrete points \( \{ x_i = (i - 1)\Delta x : i = 1, 2, \ldots, M \} \). Similarly, we partition
the time domain \([0, T]\) into \( N \) equal subintervals of length \( \Delta t \) resulting in
another set of discrete points \( \{ t_j = (j - 1)\Delta t : j = 1, 2, \ldots, N \} \). Let \( u(i, j) \)
be approximate numerical solutions of the continuous solution \( u(x, t) \) at the
nodes.

### 3.1 Derivative Approximations

We obtain approximations for the partial derivatives from Taylor series expan-
sions near the point of interest. For \( h > 0 \) and \( x_0 \in \Omega \), first derivative forward
and backward approximations are

\[
    u'(x_0) \approx \frac{u(x_0 + h) - u(x_0)}{h} + O(h) \tag{12}
\]

\[
    u'(x_0) \approx \frac{u(x_0) - u(x_0 - h)}{h} + O(h) \tag{13}
\]

respectively. Similarly we derive the second-order central difference approxi-
mation for the second derivative

\[
    u''(x_0) \approx \frac{u(x_0 + h) - 2u(x_0) + u(x_0 - h)}{h^2} + O(h^2). \tag{14}
\]

### 3.2 Forward Time, Central Space Scheme

We use the finite difference approximations developed in the preceding section
to solve (2.5) – (2.7) on discrete domain developed in section 3. Using (3.1)
and dropping the truncation error, the time derivatives in equation (2.5) can
be expressed as

\[
    \frac{\partial u(x, t)}{\partial t} = \frac{u(i, j + 1) - u(i, j)}{\Delta t}. \tag{15}
\]

Similarly, using (3.3), the spatial derivative in equation (2.5) can be expressed
as

\[
    \frac{\partial^2 u(x, t)}{\partial x^2} = \frac{u(i + 1, j) - 2u(i, j) + u(i - 1, j)}{(\Delta x)^2}. \tag{16}
\]

\[
    \frac{\partial^2 u(x^*, t)}{\partial x^2} = \frac{u(i^* + 1, j) - 2u(i^*, j) + u(i^* - 1, j)}{(\Delta x^*)^2}. \tag{17}
\]

Substituting (3.4), (3.5), and (3.6) in equation (2.5) we get

\[
    u(i, j+1) - u(i, j) = u(i^*, j+1) - u(i^*, j) \quad \frac{u(i + 1, j) - 2u(i, j) + u(i - 1, j)}{u(i^* + 1, j) - 2u(i^*, j) + u(i^* - 1, j)}. \tag{18}
\]
4 Numerical Experiments and Results

In this section, we demonstrate efficiency of numerical scheme developed in section 3 for solving problem (1.1) – (1.4). Let us define the Euclidean norm of errors at collocation points for solution $u(x, t)$ by

$$E_u = \left( \sum_{i=1}^{M} \sum_{j=1}^{N} [u_{M,N}(i, j) - u(i, j)]^2 \right)^{\frac{1}{2}}$$

(19)

where $u_{M,N}(i, j)$ is the numerical solution of $u(x, t)$ at collocation points.

For the time dependent coefficient $a(t)$, we define the relative error at collocation points by

$$E_a = \sum_{i=1}^{M} \frac{|a_{M,N}(i, j) - a(i, j)|}{|a(i, j)|}$$

(20)

where $a_{M,N}(i, j)$ is the numerical solution of $a(t)$ at collocation points.

A number of examples will be considered to verify consistency of the proposed numerical scheme. We consider the following examples.

**Example 1** : Consider (1.1) – (1.4) from section 1 with $L = 1$, $T = 1$, $x^* = 0.5$ and

$$u_0(x) = 5x^2,$$

(21)

$$\psi_1(t) = 4t + \sin(\pi t),$$

(22)

$$\psi_2(t) = 5 + 4t + \sin(\pi t),$$

(23)

$$G(t) = \frac{5}{4} + 4t + \sin(\pi t),$$

(24)

$$u(x, t) = 5x^2 + 4t + \sin(\pi t),$$

(25)

$$a(t) = \frac{2}{5} + \frac{\pi}{10} \cos(\pi t).$$

(26)
The following table compares $E_u$ and $E_a$ for different $M$, with $N = 10$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$M$</th>
<th>$E_u$</th>
<th>$E_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>50</td>
<td>4.3874 × 10^{-02}</td>
<td>2.1788 × 10^{-02}</td>
</tr>
<tr>
<td>10</td>
<td>60</td>
<td>3.7002 × 10^{-03}</td>
<td>1.2728 × 10^{-02}</td>
</tr>
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<td>70</td>
<td>2.7194 × 10^{-05}</td>
<td>1.0944 × 10^{-02}</td>
</tr>
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<td>10</td>
<td>80</td>
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<td>9.6056 × 10^{-03}</td>
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<td>90</td>
<td>2.1527 × 10^{-08}</td>
<td>8.5586 × 10^{-03}</td>
</tr>
<tr>
<td>10</td>
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<td>4.6650 × 10^{-10}</td>
<td>7.7174 × 10^{-03}</td>
</tr>
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<td>10</td>
<td>110</td>
<td>1.0509 × 10^{-12}</td>
<td>7.0269 × 10^{-03}</td>
</tr>
<tr>
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<td>120</td>
<td>4.1517 × 10^{-14}</td>
<td>6.4497 × 10^{-03}</td>
</tr>
<tr>
<td>10</td>
<td>130</td>
<td>1.9186 × 10^{-14}</td>
<td>5.9602 × 10^{-03}</td>
</tr>
</tbody>
</table>

Figure 1: Analytical solution of $a(t)$

Figure 2: Numerical solution of $a(t)$
Figure 3: Difference between numerical and analytical solutions of $a(t)$

**Example 2**: Consider (1.1) – (1.4) from section 1 with $L = 1, T = 1, x^* = 0.5$ and

\[
\begin{align*}
  u_0(x) &= x^2, \quad \psi_1(t) = 2t + \sin(\pi t) \\
  \psi_2(t) &= 1 + 2t + \sin(\pi t), \quad G(t) = \frac{1}{4} + 2t + \sin(\pi t) \\
  u(x, t) &= x^2 + 2t + \sin(\pi t), \quad a(t) = 1 + \frac{\pi}{2} \cos(\pi t) \\
\end{align*}
\]

The following table compares $E_u$ and $E_a$ for different $M$, with $N = 6$.

<table>
<thead>
<tr>
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<th>$M$</th>
<th>$E_u$</th>
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<tbody>
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<td>$4.3294 \times 10^{-06}$</td>
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<td>$1.0597 \times 10^{-07}$</td>
<td>$1.4088 \times 10^{-02}$</td>
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<td>6</td>
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<td>$3.8913 \times 10^{-10}$</td>
<td>$1.2708 \times 10^{-02}$</td>
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<td>130</td>
<td>$1.2586 \times 10^{-14}$</td>
<td>$9.8226 \times 10^{-03}$</td>
</tr>
<tr>
<td>6</td>
<td>140</td>
<td>$9.6770 \times 10^{-15}$</td>
<td>$9.1311 \times 10^{-03}$</td>
</tr>
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<td>150</td>
<td>$6.7092 \times 10^{-15}$</td>
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<td>160</td>
<td>$5.5157 \times 10^{-15}$</td>
<td>$8.0044 \times 10^{-03}$</td>
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</table>
The following table compares $E_u$ and $E_a$ for different $M$ with $N = 6$ when noise $\gamma = 0.0001$ is introduced to the overspecification data.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$M$</th>
<th>$E_u$</th>
<th>$E_a$</th>
</tr>
</thead>
<tbody>
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<td>$6.1446 \times 10^{-1}$</td>
<td>$4.1286 \times 10^{-1}$</td>
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<td>$1.9913 \times 10^{-1}$</td>
<td>$2.9335 \times 10^{-1}$</td>
</tr>
<tr>
<td>6</td>
<td>100</td>
<td>$1.5478 \times 10^{-1}$</td>
<td>$1.6975 \times 10^{-1}$</td>
</tr>
<tr>
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<td>110</td>
<td>$1.0715 \times 10^{-1}$</td>
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<td>$6.1112 \times 10^{-3}$</td>
<td>$1.1994 \times 10^{-2}$</td>
</tr>
<tr>
<td>6</td>
<td>140</td>
<td>$4.2690 \times 10^{-3}$</td>
<td>$9.6397 \times 10^{-3}$</td>
</tr>
<tr>
<td>6</td>
<td>150</td>
<td>$4.3966 \times 10^{-3}$</td>
<td>$9.0335 \times 10^{-3}$</td>
</tr>
<tr>
<td>6</td>
<td>160</td>
<td>$4.5388 \times 10^{-3}$</td>
<td>$8.5074 \times 10^{-3}$</td>
</tr>
</tbody>
</table>
Figure 6: Difference between numerical and analytical solutions of $a(t)$ with noise $\gamma = 0.0001$

The following table compares $E_u$ and $E_a$ for different $M$ with $N = 6$ when noise $\gamma = 0.001$ is introduced to the overspecification data.

<table>
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<th>$N$</th>
<th>$M$</th>
<th>$E_u$</th>
<th>$E_a$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$6.4573 \times 10^{-1}$</td>
<td>$7.3733 \times 10^{-1}$</td>
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<td>90</td>
<td>$3.5928 \times 10^{-1}$</td>
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<tr>
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<td>$1.8356 \times 10^{-1}$</td>
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<td>160</td>
<td>$4.5387 \times 10^{-2}$</td>
<td>$1.3558 \times 10^{-2}$</td>
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</tbody>
</table>
Figure 7: Difference between numerical and analytical solutions of $a(t)$ with noise $\gamma = 0.001$

The following table compares $E_u$ and $E_a$ for different $M$ with $N = 6$ when noise $\gamma = 0.01$ is introduced to the overspecification data.

<table>
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<td>$6.5260 \times 10^{-1}$</td>
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<tr>
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<tr>
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<td>120</td>
<td>$4.2331 \times 10^{-1}$</td>
<td>$1.6772 \times 10^{-1}$</td>
</tr>
<tr>
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<td>$1.2643 \times 10^{-1}$</td>
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<td>$4.5382 \times 10^{-1}$</td>
<td>$7.4300 \times 10^{-2}$</td>
</tr>
</tbody>
</table>
On the numerical solution of coefficient identification problem

Figure 8: Difference between numerical and analytical solutions of $a(t)$ with noise $\gamma = 0.01$

5 Conclusion

In this article, we used the finite difference method to identify the time dependent coefficient $a(t)$ in terms of an overspecification function $G(t)$. The Euclidean norm error of $u(x, t)$ and the relative error of $a(t)$ are compared using several examples. The errors for a fixed number of factors in space domain follow a steadily decreasing pattern. However, when a small amount of noise is introduced the error increases greatly and becomes unstable. These examples demonstrate the effectiveness of our method.

References


Received: August 7, 2014