A Single Species Model with Impulsive Diffusion and Pulsed Harvesting\textsuperscript{1}

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Abstract

In this paper, we propose a single species model with impulsive diffusion and pulsed harvesting at the different fixed time. Using the discrete dynamical system determined by the stroboscopic map, we obtain a positive fixed point and show that the positive fixed point is globally stable when harvesting for the population is less than certain value or when the period of pulsing over certain value. Correspondingly, the single species system with impulsive diffusion and pulsed harvesting always has a globally stable positive periodic solution. The main theoretical results are illustrated with numerical simulations.

Keywords: Single species model; Impulsive diffusion; Pulsed harvesting; Globally stable

1 Introduction

Because the population spatial distribution and population dynamics are greatly affected by spatial heterogeneity and population mobility, population dispersal became one of the dominant themes in mathematical ecology. In

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fact, dispersal between patches often occurs in ecological environments. Therefore, if taking the spatial heterogeneity into account, realistic models should contain dispersal process. In recent years, a lot of models are described by impulsive diffusion equations ([3, 4, 5, 7, 8, 9, 10, 12]). In these models, it is assumed that individuals in each population are identical and able to migrate to other patches. In particularly, a single population is considered in ([3, 4, 7, 8, 10, 12]). For example, Hui and Chen [4] proposed the following two patches single species impulsive diffusion system:

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t)(a_1 - b_1 x_1), \\
\dot{x}_2(t) &= x_2(t)(a_2 - b_2 x_2), \\
\triangle x_1(t) &= d_1(x_2(t) - x_1(t)), \\
\triangle x_2(t) &= d_2(x_1(t) - x_2(t)),
\end{align*}
\]

(1.1)

In this model, it is supposed that the system is composed of two patches connected by diffusion. \(x_i(t)\) represents the biomass of the population in patch \(i (i = 1, 2)\). By using the stroboscopic map, they obtained a globally stable positive fixed point of the map. Correspondingly, the single species system with impulsive diffusion always has a globally stable positive periodic solution.

In 1973 Ayala et al. [1] conducted experiments on fruit fly dynamics to test the validity of ten models of competitions. One of the models accounting best for the experimental results is given by

\[
\begin{align*}
\dot{x}_1 &= r_1 x_1 (1 - \left(\frac{x_1}{K_1}\right)^{\theta_1} - \alpha_{12} \frac{x_2}{K_2}), \\
\dot{x}_2 &= r_2 x_2 (1 - \left(\frac{x_2}{K_2}\right)^{\theta_2} - \alpha_{21} \frac{x_1}{K_1}).
\end{align*}
\]

The paper [11] also introduced a same equation to describe the growth law for a single species without exploitation, which is shown as follows:

\[
\dot{x}(t) = x(t)(a - bx^\theta(t))
\]

(1.2)

where \(x(t)\) represents the density of the resource population at time \(t\). \(a\) is the intrinsic growth rate of the resource population. \(b\) is the coefficient of intra-specific competitive. \(\theta\) provides a nonlinear measure of intra-specific interference. This equation is similar to but more general than logistic equation.

The increasing needs for food and energy lead to exploitation of renewable resources. However, to maintain sustainable productivity, biological resources should not be over exploited. We hope that sustainability could be achieved at a high level of productivity and good economic profit, which requires scientific and effective management of the biological resources. Formulating an efficient harvesting policy is complicated and difficult, but it has been studied by using operational control theory. Clark [2] supplied rich dynamic theory for renewable resource management, which constituted the beginning of a scientific theory of conservation. The optimal harvesting policy for single-species has
been systematically investigated. In this paper, the evolution of the species is modelled by the Eq. (1.2). Suppose that the population described by Eq. (1.2) is subject to harvest at rate \( h(t) = Ex(t) \), then the equation of the harvested population can be revised as following:

\[
\dot{x}(t) = x(t)(a - bx^\theta(t)) - Ex(t),
\]

where \( E \) denotes the harvesting effort. It is often the case that harvesting occurs during short-time, then the continuous harvesting of the population is removed from the model, and replaced with regular pulses.

Based on the ideas of Hui and Chen [4], we propose the following single species model with impulsive diffusion and pulsed harvesting at the different fixed time:

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t)(a_1 - b_1 x_1^{\theta_1}(t)), \\
\dot{x}_2(t) &= x_2(t)(a_2 - b_2 x_2^{\theta_2}(t)), \\
\Delta x_1(t) &= -p_1 x_1(t), \\
\Delta x_2(t) &= -p_2 x_2(t), \\
\Delta x_1(t) &= d_1 (x_2(t) - x_1(t)), \\
\Delta x_2(t) &= d_2 (x_1(t) - x_2(t)),
\end{align*}
\]

(1.3)

We suppose that system is composed of two patches connected by diffusion and occupied by a single species \( x_i (i = 1, 2) \) is the density of species in the \( i - th \) patch. \( a_i \) and \( b_i \) represent the intrinsic growth rate and density dependence rate of population in the \( i - th \) patch, respectively. \( \theta_i (i = 1, 2) \) provides a nonlinear measure of intra-specific interference in the \( i - th \) patch. \( p_i (0 \leq p_i < 1) \) is harvesting effect of the population at \( t = (n + l - 1)T, l(0 \leq l < 1) \) is positive constant. \( T \) is the period of the impulsive diffusion. The parameter \( d_i \) is the diffusion coefficient satisfying \( 0 \leq d_i < 1 \). It is assumed here that the net exchange from the \( j - th \) patch to \( i - th \) patch is proportional to difference \( x_j - x_i \) of population densities: the pulse diffusion occurs every \( T \) period. \( \Delta x_i (nT^+) = x_i(nT^+) - x_i(nT), (i = 1, 2) \) and \( x_i(nT^+) \) represents the density of population in the \( i - th \) patch immediately after the \( n - th \) diffusion pulse at time \( t = nT \), while \( x_i(nT) \) represents the density of population in the \( i - th \) patch before the \( n - th \) diffusion pulse at time \( t = nT, n = 1, 2, \ldots; a_i, b_i, \theta_i \) are positive constants.

In this paper, we concentrate on the effects of the impulsive diffusion and pulsed harvesting on the dynamics of system (1.3). The remaining of this paper is organized as follows. In section 2, the stroboscopic map (2.2) of system (1.3) is obtained, then we investigate the dynamical behaviors of system (1.3) by studying the stroboscopic map (2.2). In section 3, boundedness of map (2.2) is presented. In section 4, the global stability of the unique positive fixed point of map (2.2) is obtained by exploiting the main theorem in [6]. Most of the theoretical results are supported with numerical simulations in section 5. We conclude this paper with a brief discussion in section 6.
2 Stroboscopic Map

We can easily obtain the analytical solution of the system (1.3) at the interval $(nT, (n + 1)T]$

\[
x_i(t) = \begin{cases} 
\left( \frac{a_i}{b_i} + \frac{1}{x_i^*(nT^+)} \right) \left( \frac{1}{x_i^*(t-nT)} \right)^{\frac{1}{\eta_i}} - \frac{1}{\eta_i}, & t \in (nT, (n + l)T], \\
\left( \frac{b_i}{a_i} + \frac{1}{1-(1-p_i)\eta_i} - \frac{b_i}{a_i} \right) e^{-a_i \theta_i(t-nT)} + \frac{1}{1-(1-p_i)\eta_i} e^{-a_i \theta_i((n+1)T-t)} \right)^{-\frac{1}{\eta_i}}, & t \in ((n + l)T, (n + 1)T], i = 1, 2.
\end{cases}
\]

which yields the following stroboscopic map of system (1.3):

\[
x_1((n + 1)T^+) = \left( \frac{d_1}{w_1^1+v_1x_1^*(nT^+)} \right)^{\frac{1}{\eta_1}} + d_1 \left( \frac{d_2}{w_2^1+v_2x_2^*(nT^+)} \right)^{\frac{1}{\eta_2}},
\]

\[
x_2((n + 1)T^+) = \left( \frac{d_2}{w_2^1+v_2x_2^*(nT^+)} \right)^{\frac{1}{\eta_2}} + d_2 \left( \frac{d_1}{w_1^1+v_1x_1^*(nT^+)} \right)^{\frac{1}{\eta_1}}.
\]

here, $w_i = \frac{e^{-a_i T}}{1-p_i} > 0, v_i = \frac{b_i}{a_i} \left( 1 + \frac{1}{(1-p_i)\eta_i} - 1 \right) e^{-a_i \theta_i(T-nT)} - \frac{1}{(1-p_i)\eta_i} e^{-a_i \theta_i T} > 0$. Equations (2.2) are difference equations and describe the number of the population in two patches at pulse $t = nT$ in terms of values at $t = (n + l - 1)T$. The dynamical behavior of system (2.2) coupled with system (2.1) determines the dynamical behavior of system (1.3). Thus in the following, we will focus on system (2.2), and investigate the various dynamical behaviors.

3 Boundedness

In system (2.2), for each $i \in 1, 2$, we define the map $G_i : [0, +\infty) \to [0, +\infty)$ by $G_i(x_i) = \left( \frac{x_i^{\theta_i(nT^+)} \eta_i}{w_i^1+v_1x_i^*(nT^+)} \right)^{\frac{1}{\eta_i}}$. The iterations of the one-dimensional map $G_i$ are the density sequence generated by the single species ecological model $x_i((n + 1)T^+) = \left( \frac{x_i^{\theta_i(nT^+)} \eta_i}{w_i^1+v_1x_i^*(nT^+)} \right)^{\frac{1}{\eta_i}} (i = 1, 2)$. We denote the positive fixed point of $G_i$ by $x_i^*(i = 1, 2)$, it is easy to see that $x_i^* = \left( \frac{1-w_i^{\theta_i}}{v_i} \right)^{\frac{1}{\eta_i}} > 0$. At the same time, if $0 < x_i < x_i^*$, then $G_i(x_i) > x_i$, and if $x_i > x_i^*$, then $G_i(x_i) < x_i$. Consequently, under $G_i$ iterations, one can sees that $I_i = G_i([0, x_i^*])$ is a compact invariant interval in $R_+$ into which every point either eventually enters and stays or just limits on it.
To write system (2.2) as a map, we define the map $F : \mathbb{R}^2_+ \to \mathbb{R}^2_+$:

$$F_1(x) = \left( \frac{x_1}{w_1^{\theta_1} + v_1 x_1^{\theta_1}} \right)^{\frac{1}{\theta_1}} + d_1 \left( \frac{x_2^{\theta_2}}{w_2^{\theta_2} + v_2 x_2^{\theta_2}} \right)^{\frac{1}{\theta_2}} - \left( \frac{x_1^{\theta_1}}{w_1^{\theta_1} + v_1 x_1^{\theta_1}} \right)^{\frac{1}{\theta_1}}$$

$$F_2(x) = \left( \frac{x_2}{w_2^{\theta_1} + v_2 x_2^{\theta_1}} \right)^{\frac{1}{\theta_2}} + d_2 \left( \frac{x_1^{\theta_1}}{w_1^{\theta_1} + v_1 x_1^{\theta_1}} \right)^{\frac{1}{\theta_2}} - \left( \frac{x_2^{\theta_2}}{w_2^{\theta_2} + v_2 x_2^{\theta_2}} \right)^{\frac{1}{\theta_2}},$$

(3.1)

The set of all iterations of the map $F$ is equivalent to the set of all density sequences generated by system (2.2), $F(x)$ is the map evaluated at the point $x = (x_1, x_2) \in \mathbb{R}^2_+$. Consequently, in system (2.2), $F^n$ describes the population densities in the time $nT$.

Before stating our main results, we should firstly consider the boundedness of system (2.2). For the boundedness of system (2.2), we have the following theorem:

**Theorem 3.1** In system (2.2), every point has a bounded orbit.

**Proof** For each $i \in 1, 2$ and each point $x \in \mathbb{R}^2_+$, we need to show each of the sequence $\{F^n_i(x)\}$ is bounded. First, we show that $\max\{F_1(x), F_2(x)\} \leq \max\{G_1(x_1), G_2(x_2)\}$. If

$$G_1(x_1) = \left( \frac{x_1}{w_1^{\theta_1} + v_1 x_1^{\theta_1}} \right)^{\frac{1}{\theta_1}} \geq \left( \frac{x_2}{w_2^{\theta_2} + v_2 x_2^{\theta_2}} \right)^{\frac{1}{\theta_2}} = G_2(x_2)$$

then, from (3.1) we have

$$\max\{F_1(x), F_2(x)\} \leq \left( \frac{x_1}{w_1^{\theta_1} + v_1 x_1^{\theta_1}} \right)^{\frac{1}{\theta_1}} = G_1(x_1).$$

On the other hand, if

$$G_1(x_1) = \left( \frac{x_1}{w_1^{\theta_1} + v_1 x_1^{\theta_1}} \right)^{\frac{1}{\theta_1}} \leq \left( \frac{x_2}{w_2^{\theta_2} + v_2 x_2^{\theta_2}} \right)^{\frac{1}{\theta_2}} = G_2(x_2)$$

then, from (3.1) we have

$$\max\{F_1(x), F_2(x)\} \leq \left( \frac{x_2}{w_2^{\theta_2} + v_2 x_2^{\theta_2}} \right)^{\frac{1}{\theta_2}} = G_2(x_2).$$

Consequently, $\max\{F_1(x), F_2(x)\} \leq \max\{G_1(x_1), G_2(x_2)\}$. Recall that $I_i = G_i([0, x_i^*]) = [0, x_i^*]$ ($i = 1, 2$) are invariant sets under the iterations of $G_i$. Moreover, $H_i(x_i)$ strictly increases, if $0 < x_i < x_i^*$, then $0 < G_i(x_i) < x_i^*$. If $x_i > x_i^*$, then $G_i(x_i) < x_i$. Hence, the sequence $\{F^n_i(x)\}$ is bounded, that is, for system (2.2), every point has a bounded orbit.
4 Global Stability

The following lemma will be used to prove our main result in the next section.

Lemma 4.1 \[^6\] Let \( T : \mathbb{R}_+^n \to \mathbb{R}_+^n \) be continuous, \( C^1 \) in \( \text{int} (\mathbb{R}_+^n) \), and suppose \( DT(0) \) exists with \( \lim_{x \to 0, x > 0} DT(x) = DT(0) \). In addition, assume

(a) \( DT(x) > 0 \), if \( x > 0 \);

(b) \( DT(y) \leq DT(x) \), if \( 0 < x < y \).

If \( T(0) = 0 \), let \( \lambda = \rho(DT(0)) \). If \( \lambda \leq 1 \), then for every \( x \geq 0 \), \( T^n(x) \to 0 \) as \( n \to \infty \); if \( \lambda > 1 \) then either \( T^n(x) \to \infty \) as \( n \to \infty \) for every \( x \geq 0 \) or there exists a unique nonzero fixed point \( q \) of \( T \). In the latter case, \( q > 0 \) and for every \( x \geq 0 \), \( T^n(x) \to q \) as \( n \to \infty \).

If \( T(0) \neq 0 \), then either \( T^n(x) \to \infty \) as \( n \to \infty \) for every \( x \geq 0 \) or there exists a unique fixed point \( q \) of \( T \). In the latter case, \( q > 0 \) and for every \( x \geq 0 \), \( T^n(x) \to q \) as \( n \to \infty \).

Lemma 4.2 Suppose \( p_i < 1 - e^{-a_i T} \) or \( T > \frac{1}{a_i} \ln \frac{1}{(1-p_i)} \), if \( \frac{1-d_1}{w_1} + \frac{1-d_2}{w_2} \leq 2 \), then \( \frac{1-d_1}{w_1} + \frac{1-d_2}{w_2} > 1 + \frac{1-d_1-d_2}{w_1 w_2} \).

Proof Suppose the conclusion is not true, that is, given \( \frac{1-d_1}{w_1} + \frac{1-d_2}{w_2} \leq 1 + \frac{1-d_1-d_2}{w_1 w_2} \), it then follows that

\[
d_2 \leq (1 - w_2)(1 - \frac{d_1}{1 - w_1}).
\]

Since \( p_i < 1 - e^{-a_i T} \) or \( T > \frac{1}{a_i} \ln \frac{1}{(1-p_i)} \), and \( d_2 > 0 \), we can obtain \( d_1 < 1 - w_1 \).

The following proof is the same as Lemma 4.2 in [4]. The proof is completed.

Theorem 4.1 Suppose \( p_i < 1 - e^{-a_i T} \) or \( T > \frac{1}{a_i} \ln \frac{1}{(1-p_i)} \), there exists a unique positive fixed point \( q = (q_1, q_2) \) of the map \( F \), and for every \( x = (x_1, x_2) \geq 0 \), \( F^n(x) \to q \) as \( n \to \infty \).

Proof Let \( M = \{ g(x) \in \mathcal{C}([0, \infty), [0, \infty]) : g'(x) > 0, g''(x) < 0, g'(0) > 1, g(0) = 0, \text{ and } g(x) = x \text{ for some } x > 0 \} \), we firstly show that \( F(x) \) satisfies all the hypotheses of Lemma 4.1. Obviously, \( G_i(x_i) \in M \), so it is easy to infer that \( F(x) \) is continuous, \( C^1 \) in \( \text{int} (\mathbb{R}_+^2) \), \( F(0) = 0 \). Since

\[
DF(x, y) = \begin{pmatrix}
(1 - d_1) G'_1(x_1) & d_1 G'_2(x_2) \\
(1 - d_2) G'_1(x_2) & d_2 G'_2(x_2)
\end{pmatrix} = \begin{pmatrix}
\frac{(1-d_1)u_1^{\theta_i}}{w_1^{\theta_i + v_1 x_1}} (1+\frac{\theta_i}{\sigma_i}) & \frac{d_1 u_2^{\theta_i}}{w_2^{\theta_i + v_2 x_2}} (1+\frac{\theta_i}{\sigma_i}) \\
\frac{d_2 u_1^{\theta_i}}{w_1^{\theta_i + v_1 x_1}} (1+\frac{\theta_i}{\sigma_i}) & \frac{(1-d_2)u_2^{\theta_i}}{w_2^{\theta_i + v_2 x_2}} (1+\frac{\theta_i}{\sigma_i})
\end{pmatrix},
\]

where, \( G'_i(x_i) = \frac{u_i^{\theta_i}}{(w_i^{\theta_i + v_1 x_1})^{(1+\frac{\theta_i}{\sigma_i})}} (i = 1, 2) \), it follows that

\[
DF(0, 0) = \begin{pmatrix}
\frac{1-d_1}{d_1} & \frac{d_1}{w_1} \\
\frac{1-d_2}{d_2} & \frac{d_2}{w_2}
\end{pmatrix}.
\]
exists with \( \lim_{x \to 0, x > 0} DF(x) = DF(0) \): if \( x > 0 \), \( DF(x) > 0 \); considering

\[
G_i''(x_i) = -\frac{(1 + \theta_1)w_1^\theta_1v_1}{(w_1^\theta_1 + v_1x_1^\theta_1)^{2+\frac{1}{\theta_1}}} < 0,
\]

we know \( DF(x_1) \leq DF(x_2) \), if \( 0 < x_1 < x_2 \).

The characteristic equation of \( DF(0) \) is

\[
\lambda^2 - \left( \frac{1 - d_1}{w_1} + \frac{1 - d_2}{w_2} \right) \lambda + \frac{1 - d_1 - d_2}{w_1w_2} = 0.
\]

The following proof is the same as Theorem 4.1 in [4]. The proof is completed. \( \square \)

Theorem 4.1 shows that, for the map \( F(x) \), there exists a unique positive fixed point \( q = (q_1, q_2) \) such that \( \lim_{n \to \infty} F^n(x) = q \) for all \( x \geq 0 \), which implies the fixed point \( q = (q_1, q_2) \) of \( F \) is globally stable. So all trajectories of system (1.3) approach the positive periodic solution \((x^*_1(t), x^*_2(t))\) with period \( T \). i.e.

\[
x^*_i(t) = \begin{cases} 
\left( \frac{a_i}{b_i} + \left( \frac{1}{q_i} - \frac{b_i}{a_i} \right) e^{-a_i\theta_i(t-nT)} \right)^{-\frac{1}{\theta_i}}, & t \in (nT, (n+l)T], \\
\left( \frac{b_i}{a_i} + \frac{1}{(1-p_i)\theta_i} \left( \frac{1}{q_i} - \frac{b_i}{a_i} \right) e^{-a_i\theta_i(t-nT)} + \frac{b_i}{a_i} \left( \frac{1}{(1-p_i)\theta_i} - 1 \right) e^{-a_i\theta_i(t-(n+l)T)} \right)^{-\frac{1}{\theta_i}}, & t \in ((n+l)T, (n+1)T], \ i = 1, 2.
\end{cases}
\]

Summarizing above results, we have the following propositions:

**Theorem 4.2** Suppose \( p_i < 1 - e^{-nT} \) or \( T > \frac{1}{a_i} \ln \frac{1}{(1-p_i)} \), the fixed point \( q = (q_1, q_2) \) is globally stable, correspondingly, system (1.3) has a globally stable positive periodic solution \( x^*_i(t) \), where

\[
x^*_i(t) = \begin{cases} 
\left( \frac{a_i}{b_i} + \left( \frac{1}{q_i} - \frac{b_i}{a_i} \right) e^{-a_i\theta_i(t-nT)} \right)^{-\frac{1}{\theta_i}}, & t \in (nT, (n+l)T], \\
\left( \frac{b_i}{a_i} + \frac{1}{(1-p_i)\theta_i} \left( \frac{1}{q_i} - \frac{b_i}{a_i} \right) e^{-a_i\theta_i(t-nT)} + \frac{b_i}{a_i} \left( \frac{1}{(1-p_i)\theta_i} - 1 \right) e^{-a_i\theta_i(t-(n+l)T)} \right)^{-\frac{1}{\theta_i}}, & t \in ((n+l)T, (n+1)T], \ i = 1, 2.
\end{cases}
\]

5 **Numerical simulations**

To demonstrate the theoretical results obtained in this paper, we will give some numerical simulations. We consider the hypothetical set of parameter values as

\[
a_1 = 2, b_1 = 0.6, a_2 = 2, b_2 = 1, \theta_1 = 0.5, \theta_2 = 0.3, p_1 = 0.2, p_2 = 0.5, d_1 = 0.3, d_2 = 0.6
\]
By directly computing, we obtain $p_1 < 1 - e^{-a_1 T} \approx 0.865$ or $T > \frac{1}{a_1} \ln \frac{1}{1-p_1} \approx 0.112$, $p_2 < 1 - e^{-a_2 T} \approx 0.865$ or $T > \frac{1}{a_2} \ln \frac{1}{1-p_2} \approx 0.347$. According to Theorem 4.2, we know that system (1.3) has a globally stable positive periodic solution $x_i(t)$ for this case (see Fig. (a)-(c)).

![Time series plots](image)

Fig. (a)-(c) show that system (1.3) with initial condition has $(x_1(0^+), x_2(0^+)) = (6.6, 3.75)$ has a positive periodic solution.

### 6 Conclusions

The focus of this paper is the dynamic of a single species model with impulsive diffusion and pulsed harvesting at the different fixed time. By using the stroboscopic map, we have obtained a positive fixed point of the map (2.2). From Theorem 4.2, if $p_i < 1 - e^{-a_i T}$ or $T > \frac{1}{a_i} \ln \frac{1}{1-p_i}$, then the map (2.2) exists a globally stable positive fixed point. Correspondingly, system (1.3) has a globally stable positive periodic solution. We can see that a small pulse catching rate (with $p_i$) or a long period of pulsing (with $T$) is sufficient condition for the single species survive in the two patches. A homologous result was also given by [4] if $p_i = 0$ and $\theta_i = 1$.

### References


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