L-Nesting and Multinesting in G-Designs
Recent Results and 50 New Open Problems

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Abstract

This paper contains a survey of recent results about nesting and multinesting in G-designs, with generalizations. In the last section there is a list of 50 open problems, useful for next possible research.

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1 Introduction

A G-design of order v and index λ is a pair Σ=(X,B), where X is a set of v distinct elements (vertices) and B is a family of graphs (blocks), each isomorphic to G, such that for every pair of distinct vertices x,y ∈ X, there exist exactly λ graphs of B having the pair {x,y} as an edge. A G-design is also called a G-decomposition of λK_v. In the literature of Design Theory there is a lot of papers about the concept of nesting. In this paper we give a survey about nesting in G-designs and their recent generalizations, recalling the definitions in historical order. At last, we give a list of interesting 50 open problems.

In what follows, we will denote by C_m a cycle with m vertices, by S_m a star having a center and m pendant vertices and by K_v the complete graph with v vertices.
2 Nesting

If \( \Sigma = (X, C) \) is a \( C_m \)-design of order \( v \) and index one, a nesting of \( \Sigma \) is a mapping \( \varphi: C \rightarrow X \) such that the collection of all the stars

\[
\Pi = \{ \{x, \varphi(c)\} : c \in C, x \in c, \varphi(c) \notin c \}
\]

is an \( S_m \)-design, i.e. an \( S_m \)-partition of \( K_v \).

This definition of nesting was first introduced for \( C_m \)-designs by C.C.Lindner, C.Rodger, C.Colbourn, and R.Stinson, and was studied by many authors. At first, it was examined the determination of the spectrum of \( C_m \)-designs having a nesting for \( m = 3 \) in [8] and after in [19,20], where 15 possible exceptions remained. In [24,25] the author completed the spectrum, studying also the case \( m = 5 \).

3 L-Nesting and LG-nesting

Observe that the definition above requires that to determine a nesting for a \( G \)-design, where \( G = (V, E) \), it is necessary that \( |V| = |E| \).

A generalization of nesting for \( G \)-designs that required \( |V| \geq |E| \) was introduced by S.Milici and G.Quattrocchi in [22,23].

The following is a new and more general definition of nesting, introduced by L.Gionfriddo in [9,10], in which there are no restrictions between the number of vertices and the number of edges of graph \( G \). We will call it \( L \)-nesting for \( G \)-designs.

**Definition 1**: Let \( G=(V_G, E_G) \), \( H=(V_H, E_H) \) be two graphs and let \( \Sigma = (X, B) \) be a \( G \)-design of index \( \lambda \). A nesting \( N(G, H; \lambda, \mu) \) of \( \Sigma \) is a triple \( N=(\Sigma, \Pi, F) \), where \( \Pi=(V_H, S) \) is an \( m \)-star-design of index \( \mu \) and \( F: B \rightarrow S \) is a bijection such that:

1. for every \( B \in B \), the center of the \( m \)-star \( F(B) \) is not a vertex of \( B \);
2. for every \( B \in B \), \( x \) is a terminal vertex of \( F(B) \) if and only if \( x \) is a vertex of \( B \);
3. for every pair of blocks \( B_1, B_2 \in B \), the graphs \( B_1 \cup F(B_1) \) and \( B_2 \cup F(B_2) \) are isomorphic.

If \( H \) is isomorphic to \( K_v \), then the nesting will be denoted by \( N(G, v; \lambda, \mu) \).

**Example 1**: If by \( [x; x_1,x_2,..., x_k] \) we denote the union-graph between the path \( P_k \) having vertices \( x_1,x_2,...,x_k \) and edges \( \{x_1,x_2\},\{x_2,x_3\},...,\{x_{k-1},x_k\} \) and
the star having center \( x \) and terminal vertices \( x_1, x_2, \ldots, x_k \), then the design defined on \( \mathbb{Z}_5 \) having for blocks \( \{ j; j+1, j+2, j+3, j+4 \} \), \( \{ j; j+2, j+4, j+1, j+3 \} \) for every \( j = 0, 1, 2, 3, 4 \), is the nesting \( N(P_4; 5, 3, 4) \).

Some necessary conditions follow.

**Theorem 3.1** Let \( G = (V, E) \) be a graph and \( \Sigma = (X, B) \) be a \( G \)-design of index \( \lambda \). If there exists an \( N(G; n; \lambda, \mu) \), then \( \lambda \cdot |V| = \mu \cdot |E| \).

**Theorem 3.2** For a nested design \( N(P_k; n; \lambda, \mu) \), necessarily:

i) \( \lambda = (k - 1) \cdot \rho \) and \( \mu = k \cdot \rho \) for some \( \rho \in \mathbb{N} \);

ii) if \( n = k + 1 \), then \( k \cdot \rho \) is an even number.

The following results about \( L \)-nesting for \( P_k \)-designs have been obtained in [9].

**Theorem 3.3** Let \( k \) be an integer, \( k > 1 \). For every prime number \( n > k \), there exists a nested design \( N(P_k; n; \lambda, \mu) \).

**Theorem 3.4** For every prime number \( n \in \mathbb{N} \) and for every graph \( G = (V, E) \) containing a hamiltonian path, there exists a nested \( G \)-design \( N(G; n; \lambda, \mu) \) with \( \lambda = |E|, \mu = |V| \).

**Theorem 3.5**

1. For any prime number \( n \in \mathbb{N} \), there exists a nested design \( N(K_k; n; \lambda, \mu) \) with \( \lambda = \binom{k}{2}, \mu = k \).

2. There exists a nested design \( N(G; n; \nu_1, \nu_2) \), where \( G \) is a spanning subgraph of \( K_k \) and \( \nu_1 = |E|, \nu_2 = |V| = k \).

**Theorem 3.6** If there exists a nested design \( N(C_m; n; 1, 1) \), then for every \( k, 3 \leq k < m \), there exists a nested design \( N(P_k; n; 1, 1) \).

In [11] the following concept was introduced.

**Definition 2:** Let \( G = (V, E) \) be a graph and let \( \Sigma = (X, B) \) be a \( G \)-design of index \( \lambda_1 \) and order \( n \). Let \( h \) be an integer such that: \( 1 \leq h \leq n - |V| \).

The \( LG(1) \)-nesting of \( \Sigma \) is the nesting \( N_1 = N(G; n; \lambda_1, \mu_1) = (\Sigma, \Pi, F) \) introduced in Definition 1.

Let \( G_1 = G, \Pi_1 = \Pi, F_1 = F \). For \( h \geq 2 \) the \( LG(h) \)-nesting of \( \Sigma \) is the nesting \( N_h = (\Sigma_h, \Pi_h, F_h) \) of \( \Sigma_h = N(G_{h-1}; n; \lambda_{h-1}, \mu_{h-1}) \).

We say such a nesting \( LG(h) \)-nesting.
We precise that in what follows, if $B \in \mathcal{B}$ and $x$ is the centre of $F(B)$, we will write $[x; B]$ instead of $B \cup F(B)$.

Results:

**Theorem 3.7** - Let $G_1=(V_1,E_1)$ be a graph and let $\Sigma_1 = (X, B_1)$ be a $G_1$-design of index $\lambda_1$. A necessary condition for the existence of a $N(G_h,n; \lambda_h, \mu_h)$, for $h = 1, 2, \ldots, n - |V_1|$, is that for every $i = 1, 2, \ldots, h$: $\lambda_i \cdot (|V_1 + i - 1|) = \mu_i \cdot (|E_1 + (i - 1) \cdot (|V_1| + \frac{i-2}{2}))$.

**Theorem 3.8** Let $N=(\Sigma, \Pi, F)$ be a nested design $N(P_k,n; \lambda, \mu)$. Then, necessarily:

i) $\lambda_1 = (k - 1) \cdot \rho, \quad \mu_1 = k \cdot \rho$, for some positive integer $\rho$;

ii) if $n = k + 1$, then $k \cdot \rho$ is an even number.

**Theorem 3.9** Let $k$ be an integer, $k>1$. For every prime number $n$, $n > k$, there exists an $LG(h)$-nested design $N(P_k,n; \lambda_h, \mu_h)$, for every $h$ such that: $1 \leq h \leq n - |V(G)|$.

For other details see [11].

4 Multinesting

In [2,3,4] the following general definition of nesting is given.

**Definition 3**: Let $G_1=(V_1,E_1)$, $G_2=(V_2,E_2)$, $G=(V,E)$ be graphs, with $G_1, G_2$ subgraphs of $G$ such that $E_1$ and $E_2$ form a partition of $E$. Further, let $\Sigma = (X, B)$ be a $G$-design of index $\lambda$. A nesting $N(G,G_1; \lambda, \mu)$ of $\Sigma$ is a pair $N=(\Sigma, \Pi)$, where $\Pi=(X, B_1)$ is a $G_1$-design of index $\mu$ having for blocks all the $G_1$ graphs obtained from the blocks of $\Sigma$ taking for everyone the subgraph isomorphic to $G_1$.

Observe that there exists a bijection $F$: $\mathcal{B} \rightarrow \mathcal{B}_1$ which associates with every block of $\Sigma$ the correspondent subgraph block of $\Pi$. Further, also the subgraphs $G_2$ of the blocks of $\Sigma$ form a $G_2$-design of index $\lambda - \mu$.

Often, in the situation seen above, $\Sigma$ is said to be $G_1$-perfect [2,3,4,18,21].

Similarly, it is possible to consider multinestings [15].
**Definition 4:** Let \( G_1=(V_1,E_1), G_2=(V_2,E_2), \ldots, G_k=(V_k,E_k), G=(V,E) \) be graphs, with \( G_1, G_2, \ldots, G_k \) subgraphs of \( G \), such that \( E_1, E_2, \ldots, E_k \) have no edges in common. Let \( \Sigma=(X,B) \) be a \( G \)-design of index \( \lambda \). A multinesting \( N(G,G_1,\ldots,G_k;\lambda,\mu_1,\ldots,\mu_k) \) of \( \Sigma \) is an ordered collection \( N=(\Sigma,\Pi_1,\ldots,\Pi_k) \), where for every \( i=1,2,\ldots,k \), \( \Pi_i=(X,B_i) \) is a \( G_i \)-design of index \( \mu_i \), having for blocks all the \( G_i \) graphs obtained from the blocks of \( \Sigma \) taking for everyone the subgraph isomorphic to \( G_i \).

Observe that, also in these cases, there exist \( k \) bijections \( F_1,F_2,\ldots,F_k \), \( F_i: B \to B_i \), for every \( i=1,2,\ldots,k \), which associate with every block of \( \Sigma \) the correspondent subgraph block of \( \Pi_i \).

**4.1) - The case of Octagon Quadrangle System**

An octagon quadrangle \( OQ \) is a graph \( G=(V,E) \), having vertex-set \( V=\{x_1,x_2,\ldots,x_8\} \), and edge-set

\[
E = \{\{x_i,x_{i+1}\} : i = 1,2,\ldots,7\} \cup \{\{x_1,x_8\}, \{x_1,x_4\}, \{x_5,x_8\}\}.
\]

In other words, \( OQ \) is the graph obtained from a cycle of length eight adding two parallel edges which divide the octagon in three quadrangles. Such a graph will be denoted by \([x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8]\). The cycle \((x_1,x_4,x_5,x_8)\) will be the inside \( C_4 \)-cycle, while the cycle \((x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\) will be the outside cycle [2].

An octagon quadrangle system of order \( v \) and index \( \lambda \), briefly an \( OQS(v) \), is a pair \( \Sigma=(X,B) \), where \( X \) is a finite set of \( v \) vertices and \( B \) is a collection of edge disjoint octagon quadrangles, called blocks, which partitions the edge set of \( \lambda K_v \), defined on the vertex set \( X \).

Following the definitions given above, an octagon quadrangle system \( \Sigma=(X,B) \) of order \( v \) and index \( \lambda \) is said to be:

1) \( C_4 \)-perfect, if all of the inside \( C_4 \)-cycles contained in the octagon quadrangles form a \( \mu \)-fold 4-cycle system of order \( v \);

2) \( C_8 \)-perfect, if all of the outside \( C_8 \)-cycles contained in the octagon quadrangles form a \( \varrho \)-fold 8-cycle system of order \( v \);

3) strongly perfect, if the collection of all of the inside \( C_4 \)-cycles contained in the octagon quadrangles form a \( \mu \)-fold 4-cycle system of order \( v \) and the collection of all of the outside \( C_8 \)-cycles contained in the octagon quadrangles
form a $\varrho$-fold 8-cycle system of order $v$.

In the first two cases, we say that the system has indices $(\lambda, \mu)$ or $(\lambda, \varrho)$ respectively, in the third case we say that the system has indices $(\lambda, \varrho, \mu)$. Also, in $i$) the $C_4$-system is nested in the OQS, in $ii$) the $C_8$-system is nested in the OQS, in both cases the OQS is nesting the $C_4$-system and the $C_8$-system.

In the following examples there are OQSs of different types. The vertex set is always $Z_v$. The collection $\mathcal{B}$ of octagon quadrangles is given by a set of base blocks. If $B^* = [(a, b, c, (d), (\alpha), (\beta, \gamma, (\delta))]$ is a base block, then $B^*_i = [(a + i), b + i, c + i, (d + i), (\alpha + i), (\beta + i, \gamma + i, (\delta + i)]$ is a block of $B$, for each $i = 1, 2, ..., v \in Z_v$. $B^*_i$ is said to be a translated block of $B^*$.

**Example 2:** The following blocks define a strongly perfect OQS(13) of indices $(5,4,2)$. The inside $C_4$-cycles form a $C_4$-system of index $\mu = 2$ and the outside $C_8$-cycles form a $C_8$-system of index $\varrho = 4$.

*Base blocks* (mod 13):

\[[(0), 5, 9, (3), (7), 4, 2, (1)],\]
\[[(0), 10, 5, (1), (7), 6, 8, (2)],\]
\[[(0), 6, 10, (2), (7), 4, 5, (3)].\]

**Example 3:** The following blocks define a $C_4$-perfect OQS(13) of indices $(5,2)$. The inside $C_4$-cycles form a $C_4$-system of index $\mu = 2$; the outside $C_8$-cycles do not form a $C_8$-system.

*Base blocks* (mod 13):

\[[(1), 9, 11, (2), (5), 3, 7, (8)],\]
\[[(3), 4, 0, (10), (5), 11, 8, (1)],\]
\[[(8), 0, 6, (9), (1), 2, 10, (12)].\]

**Example 4:** The following blocks define a $C_8$-perfect OQS(13) of indices $(5,4)$. The outside $C_8$-cycles form a $C_8$-system of index $\varrho = 4$; the inside $C_4$-cycles do not form a $C_4$-system.

*Base blocks* (mod 13):

\[[(0), 8, 10, (1), (4), 2, 6, (7)].\]
Example 5: The following blocks define an OQS(13) of index $\lambda = 5$. It is not perfect. The outside $C_8$-cycles do not form a $C_8$-system and the inside $C_4$-cycles do not form a $C_4$-system.

*Base blocks* (mod 13):

\[
[(0), 10, 4, (1), 6, 7, 8, (4)],$

\[
[(2), 8, 5, (3), 1, 10, 9, (4)],$

\[
[(3), 5, 11, (6), 1, 8, 12, (7)].$

In [2,4] the spectrum is determined for many classes of OQSs, as we can see in what following. This research follows the results found in [12,13] for hexagon quadrangle systems and hexagon kite systems.

**Theorem 4.1**: There exist strongly perfect OQS(v)s of indices (5, 4, 2) if and only if $v \equiv 0$ or 1 mod 8, $v \geq 8$.

**Theorem 4.2**: There exist OQS(v)s of indices (5, 2), which are $C_4$-perfect but not $C_8$-perfect, if and only if $v \equiv 0$ or 1 mod 8, $v \geq 8$.

**Theorem 4.3**: There exist OQS(v)s of indices (5, 4), which are $C_8$-perfect but not $C_4$-perfect, if and only if $v \equiv 0$ or 1 mod 8, $v \geq 8$.

**Theorem 4.4**: There exist OQS(v)s of index 5, which are neither $C_4$-perfect nor $C_8$-perfect, if and only if $v \equiv 0$ or 1 mod 8, $v \geq 8$.

4.2) - A case of Multinesting

For Octagon Quadrangle Systems there are also the following definitions. An OQS $\Sigma = (X, B)$ of order $v$ and index $\lambda$ is said to be [3]:

4) upper $C_4$-perfect, if all of the upper $C_4$-cycles contained in the octagon quadrangles form a $\mu_1$-fold 4-cycle system of order $v$;
5) **lower C\textsubscript{4}-perfect**, if all of the lower C\textsubscript{4}-cycles contained in the octagon quadrangles form a \(\mu_2\)-fold 4-cycle system of order \(v\);

6) **super-perfect**, if \(\Sigma\) is upper, lower and outside perfect and in this case we say also that \(\Sigma\) is a **total nesting system**.

In every case, the system \(\Sigma'\) contained in \(\Sigma\) is said to be **nested** in it and \(\Sigma\) is said **nesting** \(\Sigma'\).

We give some examples. In them the vertex set is always \(Z_v\) or \(Z_{v-1}\cup\{\infty\}\). We use **base blocks** and the symbol \(\infty\), with the condition that \(\infty + i = \infty\), for every positive integer \(i\).

**Example 6:** The following blocks define an \(OQS(17)\) of indices \((5,4,2)\), which is upper-C\textsubscript{4} perfect and C\textsubscript{8} perfect. We can see that the upper C\textsubscript{4}-cycles form a \(C_4\)-system of index \(\mu = 2\) and the outside C\textsubscript{8}-cycles form a \(C_8\)-system of index \(\varphi = 4\). Observe that the lower C\textsubscript{4}-cycles do not form a \(C_4\)-system, this \(OQS\) is **not strongly perfect**.

**Base blocks** (mod 17):

\[
[(0), 14, 15, (6), (12), 7, 5, (13)],

[(0), 13, 1, (8), (10), 9, 11, (7)],

[(0), 13, 1, (2), (11), 4, 16, (6)],

[(0), 3, 9, (7), (10), 2, 5, (6)].
\]

**Example 7:** The following blocks define a **super-perfect** \(OQS(13)\) of indices \((5,4,2)\). The upper C\textsubscript{4}-cycles form a \(C_4\)-system of index \(\mu = 2\); the lower C\textsubscript{4}-cycles form another \(C_4\)-system of index \(\mu = 2\); the outside C\textsubscript{8}-cycles form a \(C_8\)-system of index \(\varphi = 4\). There are three cycles-systems nested in this \(OQS(13)\).

**Base blocks** (mod 13):

\[
[(0), 3, 10, (1), (7), 9, 4, (2)],

[(0), 1, 10, (2), (7), 8, 4, (3)],

[(0), 2, 10, (3), (7), 4, 11, (1)].
\]

**Example 8:** Let \(\Sigma = (Z_9, \mathcal{B})\) be the system defined in \(Z_9\) whose blocks are
all the translates obtained by the following:

**Base blocks** (mod 9):

\[(0, 1, 5, 7, 4, 3, 6, 8),
(3, 0, 5, 2, 4, 8, 6, 7)\].

We can verify that \(\Sigma\) is a **super-perfect OQS(9)** of indices (5,4,2,2). The **upper** \(C_4\)-system is generated by the two base 4-cycles:

\[(0, 1, 5, 7), (3, 0, 5, 2)\].

The **lower** \(C_4\)-system is generated by the two base 4-cycles:

\[(4, 3, 6, 8), (4, 8, 6, 7)\].

**Example 9**: Let \(\Sigma = (X, B)\) be the system defined in \(X = \mathbb{Z}_7 \cup \{\infty\}, \infty \notin \mathbb{Z}_7\), whose blocks are all the translates obtained by the following:

**Base blocks** (mod 7):

\[(\infty, 5, 6, 3, 2, 0, 1, 4),
(1, 0, 2, 4, 6, 3, \infty, 5)\],

where \(\infty\) is a fixed vertex and all the others are obtained cyclically in \(\mathbb{Z}_7\). We can verify that \(\Sigma\) is a **super-perfect OQS(8)** of indices (5,4,2,2). The **upper** \(C_4\)-system is generated by the two base 4-cycles:

\[(\infty, 5, 6, 3), (1, 0, 2, 4)\].

The **lower** \(C_4\)-system is generated by the two base 4-cycles:

\[(2, 0, 1, 4), (\infty, 5, 6, 3)\].

**Example 10**: Let \(\Sigma = (X, B)\) be the system defined in \(X = \mathbb{Z}_{11} \cup \{\infty\}, \infty \notin \mathbb{Z}_{11}\), whose blocks are all the translated one obtained by the following:

**Base blocks** (mod 11):

\[(\infty, 10, 8, 5, 6, 7, 9, 1),
(0, 3, 8, 1, 6, 7, 9, 2),
(0, 1, 8, 2, 10, 5, \infty, 7)\],

where \(\infty\) is a fixed vertex and all the others are obtained cyclically in \(\mathbb{Z}_{11}\). We can verify that \(\Sigma\) is a **super-perfect OQS(12)** of indices (5,4,2,2). The **upper** \(C_4\)-system is generated by the 4-cycles:
The lower $C_4$-system is generated by the 4-cycles:
$(1,6,7,9), (2,6,7,9), (∞, 7, 10, 5)$.

The following are the main results obtained in [15] about multnestings in $OQS$, also called super-perfect $OQS$.

**Theorem 4.5**: There exist super-perfect $OQS(v)$ of indices $(5,4,2,2)$ if and only if $v ≡ 0$ or $1$ mod $4$, $v ≥ 8$.

**Theorem 4.6**: For every $v ≡ 0$ or $1$ mod $4$, $v ≥ 8$, there exist $OQS$s of order $v$ and index 5 nesting two $C_4$-systems of index 2 and a complete graph $K_v$.

**Theorem 4.7**: For every $v ≡ 1$ mod $8$, $v ≥ 9$, there exist $OQS$s of order $v$ and index 5 nesting a $C_4$-system of index 5 decomposable into two $C_4$-systems of index 2 and a $C_4$-system of index one.

### 4.3) - The case of Dodecagon Quadrangle Systems

A dodecagon quadrangle is a graph $G = (V, E)$, having vertex-set $V = \{x_1, x_2, ..., x_{12}\}$ and edge-set

$$E = \{\{x_i, x_{i+1}\} : i = 1, 2, ..., 11\} \cup \{\{x_1, x_{12}\}, \{x_1, x_4\}, \{x_4, x_7\}, \{x_7, x_{10}\}, \{x_1, x_{10}\}\}$$

and it will be denoted by $[(x_1), x_2, x_3, (x_4), x_5, x_6, (x_7), x_8, x_9, (x_{10}), x_{11}, x_{12}]$.

A dodecagon quadrangle system of order $v$ and index $\rho$, briefly a $DQS$, is a pair $(X, \mathcal{H})$, where $X$ is a finite set of $v$ vertices and $\mathcal{H}$ is a collection of edge disjoint dodecagon quadrangles (called blocks) which partitions the edge set of $\rho K_v$, with vertex set $X$.

A dodecagon quadrangle system $(X, \mathcal{H})$ of order $v$ and index $\rho$ is said to be perfect, briefly a $PDQS$ if the collection of all of the inside 4-cycles contained in the dodecagon quadrangles form a $\mu$-fold 4-cycle system of order $v$. Usually, this $\mu$-fold 4-cycle system is also said nested in the $DQS$ $(X, \mathcal{H})$ [14].

We can observe that in a $PDQS$ the inside 4-cycles contained in the blocks form a 4-cycle system which is nested in the $DQS$.

In the following examples the vertex set is always $Z_{33}$.

**Example 11**: The following system is a $DQS(33)$ of index one, but it is not a PDQS.
Base block (mod 33):

\[ [(0), 11, 21, (1), 7, 14, (18), 6, 8, (3), 17, 9]. \]

**Example 12:** The following system is a \( PDQS(33) \) having index four. The inside 4-cycle system has index one.

Base blocks (mod 33):

\[ [(0), 32, 7, (1), 4, 20, (17), 18, 31, (8), 13, 29], \]
\[ [(0), 20, 11, (2), 13, 9, (17), 24, 31, (7), 8, 19], \]
\[ [(0), 14, 29, (3), 7, 9, (17), 5, 11, (6), 28, 31], \]
\[ [(0), 16, 22, (4), 9, 19, (17), 30, 15, (5), 26, 14]. \]

In [14] it is determined completely the spectrum of \( DQSs \) and the spectrum of \( PDQSs \).

Results:

**Theorem 4.8:** There exists a \( DQS \) of order \( v \) and index one if and only if \( v \equiv 1 \mod 32, n \geq 33 \).

**Theorem 4.9:** There exists a \( PDQS \) of order \( v \) and minimum index if and only if \( v \equiv 1 \mod 8, v \geq 17 \).

To prove these Theorems, the following interesting constructions are used:

1) **Construction** \( v \to v + 32 \) for \( DQSs \)

Let \( Z_{9,i} = Z_9 \times \{i\} \), for \( i=1, 2, \ldots, 4k, a, b, c, d \) (all distinct elements), where \( (0, i) = 0 \) for every \( i=1, 2, \ldots, 4k, a, b, c, d \). Further, let \( (x, i) = x_i \). Let \( (A, S_1) \) be a \( DQS(v) \) of order \( v = 32k + 1, v \geq 33 \) and let \( (B, S_2) \) be a \( DQS(33) \) of order 33, where:

\[ A = \bigcup_{i=1}^{4k} Z_{9,i}, \text{ and } \]
\[ B = \bigcup_{i=1}^{4k} Z_{9,i}. \]
Define in \( A \cup B \) the family \( \mathcal{H}^* \) of dodecagon quadrangles such that: \( \mathcal{H}_1 \subseteq \mathcal{H}^* \), \( \mathcal{H}_2 \subseteq \mathcal{H}^* \).

Further, for every \( i=1, 2, \ldots, 4k \) and for every \( j=a, b, c, d \), if:

\[
\Phi(1)_{i,j} = \{[(1_j), 5_i, 7_j, (3_i), 8_j, 7_i, (2_j), 6_i, 4_j, (1_i), 3_j, 2_i]\}
\]

\[
\Phi(2)_{i,j} = \{[(3_j), 6_i, 1_j, (4_i), 2_j, 2_i, (4_j), 8_i, 6_j, (3_i), 5_j, 5_i]\}
\]

\[
\Phi(3)_{i,j} = \{[(3_j), 4_i, 5_j, (6_i), 6_j, 2_i, (4_j), 5_j, 2_j, (8_i), 1_j, 7_i]\}
\]

\[
\Phi(4)_{i,j} = \{[(5_j), 8_i, 3_j, (7_i), 4_j, 5_i, (6_j), 4_i, 8_j, (1_i), 7_j, 2_i]\}
\]

then:

\[
\Phi(1)_{i,j} \subseteq \mathcal{H}^*, \Phi(2)_{i,j} \subseteq \mathcal{H}^*,
\]

\[
\Phi(3)_{i,j} \subseteq \mathcal{H}^*, \Phi(4)_{i,j} \subseteq \mathcal{H}^*.
\]

If \( X = A \cup B \) and \( \mathcal{H}^* = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \Phi(1)_{i,j} \cup \Phi(2)_{i,j} \cup \Phi(3)_{i,j} \cup \Phi(4)_{i,j} \) then, examining by difference methods that every pair of distinct elements of \( X \) is contained in exactly one dodecagon quadrangle of \( \mathcal{H}^* \), it is straightforward to verify that \((X, \mathcal{H}^*)\) is a DQS of order \( v + 32 \).

We observe that the number of blocks of \( \mathcal{H}^* \), counting \( \Phi(u)_{i,j} \) for every \( i=1, 2, \ldots, 4k \) and for every for every \( j=a, b, c, d \), is:

\[
|\mathcal{H}^*| = |\mathcal{H}_1| + |\mathcal{H}_2| + |\Phi(1)_{i,j}| + |\Phi(2)_{i,j}| + |\Phi(3)_{i,j}| + |\Phi(4)_{i,j}| =
\]

\[
\left(\frac{32k+1}{16}\right) + \left(\frac{33}{3}\right) + 64k = 32k^2 + 65k + 33,
\]

which is exactly the number of blocks of a DQS(32k + 33):

\[
\left(\frac{32k+33}{16}\right) = 32k^2 + 65k + 33,
\]

and the construction is completed. \( \square \)

2) Construction \( v \rightarrow v + 16 \) for PDQSs

Let \( Z_{4k,i} = Z_{4k} \times \{i\}, i=1,2 \), and let \( Z_{8,j} = Z_8 \times \{j\}, j=a,b \). Further, let \( (x, i) = x_i \).

Let \((A, H_1)\) be a PDQS\((v)\) of order \( v = 8k + 1 \), \( v \geq 17 \), and let \((B, H_2)\) be a
PDQS(17) of order 17, where:

\[ A = Z_{4k,1} \cup Z_{4k,2} \cup \{\infty\} \quad \text{and} \quad B = Z_{8,a} \cup Z_{8,b} \cup \{\infty\}. \]

Define on \( A \cup B \) the family \( \mathcal{H}^* \) of dodecagon quadrangles such that: \( \mathcal{H}_1 \subseteq \mathcal{H}^* \), \( \mathcal{H}_2 \subseteq \mathcal{H}^* \). Further, for every \( i \in \mathbb{Z}_{4k} \), let:

\[
\begin{align*}
\Delta(1)_a &= \{[(i + 1)_1, 4_a, (i + 2)_1, (2_a), (i + 2)_2, 8_a, ((i + 1)_2), 7_a, (i + 3)_2, (1_a), (i + 3)_1, 3_a]\}, \\
\Delta(2)_a &= \{[(i + 1)_1, 6_a, (i + 2)_1, (4_a), (i + 2)_2, 2_a, ((i + 1)_2), 1_a, (i + 3)_2, (3_a), (i + 3)_1, 5_a]\}, \\
\Delta(3)_a &= \{[(i + 1)_1, 8_a, (i + 2)_1, (6_a), (i + 2)_2, 4_a, ((i + 1)_2), 3_a, (i + 3)_2, (5_a), (i + 3)_1, 7_a]\}, \\
\Delta(4)_a &= \{[(i + 1)_1, 2_a, (i + 2)_1, (8_a), (i + 2)_2, 6_a, ((i + 1)_2), 5_a, (i + 3)_2, (7_a), (i + 3)_1, 5_a]\}, \\
\Delta(1)_b &= \{[(i + 1)_1, 4_b, (i + 2)_1, (2_b), (i + 2)_2, 8_b, ((i + 1)_2), 7_b, (i + 3)_2, (1_b), (i + 3)_1, 3_b]\}, \\
\Delta(2)_b &= \{[(i + 1)_1, 6_b, (i + 2)_1, (4_b), (i + 2)_2, 2_b, ((i + 1)_2), 1_b, (i + 3)_2, (3_b), (i + 3)_1, 5_b]\}, \\
\Delta(3)_b &= \{[(i + 1)_1, 8_b, (i + 2)_1, (6_b), (i + 2)_2, 4_b, ((i + 1)_2), 3_b, (i + 3)_2, (5_b), (i + 3)_1, 7_b]\}, \\
\Delta(4)_b &= \{[(i + 1)_1, 2_b, (i + 2)_1, (8_b), (i + 2)_2, 6_b, ((i + 1)_2), 5_b, (i + 3)_2, (7_b), (i + 3)_1, 5_b]\}. 
\end{align*}
\]

Then:

\[ \Delta(1)_a \subseteq \mathcal{H}^*, \Delta(2)_a \subseteq \mathcal{H}^*, \Delta(3)_a \subseteq \mathcal{H}^*, \Delta(4)_a \subseteq \mathcal{H}^*, \quad \text{and} \]

\[ \Delta(1)_b \subseteq \mathcal{H}^*, \Delta(2)_b \subseteq \mathcal{H}^*, \Delta(3)_b \subseteq \mathcal{H}^*, \Delta(4)_b \subseteq \mathcal{H}^*. \]

If \( X = A \cup B \) and
\[ H^* = H_1 \cup H_2 \cup \Delta(1)_a \cup \Delta(2)_a \cup \Delta(3)_a \cup \Delta(4)_a \cup \Delta(1)_b \cup \Delta(2)_b \cup \Delta(3)_b \cup \Delta(4)_b \]

then, examining by difference methods that every pair of distinct elements of \( X \) is contained in exactly four \textit{dodecagon quadrangles} of \( H^* \) and that the \textit{inside} 4-cycles form a 4-cycle system of the same order and index one, it is straightforward to verify that \((X, H^*)\) is a \textit{PDQS} of order \( v + 16 \) and index four (minimum possible).

We observe that the number of blocks of \( H^* \) is:

\[
|H|^* = |H_1| + |H_2| + |\Delta(1)_a| + |\Delta(2)_a| + |\Delta(3)_a| + |\Delta(4)_a| + |\Delta(1)_b| + |\Delta(2)_b| + |\Delta(3)_b| + |\Delta(4)_b| = \left( \frac{8k + 1}{2} \right) 4 + \left( \frac{17}{16} \right) 4 + 32k = 8k^2 + 33k + 34,
\]

which is exactly the number of blocks of a \textit{PDQS} \((8k + 17)\) of index four:

\[
\left( \frac{8k + 17}{16} \right) 4 = 8k^2 + 33k + 34,
\]

and the construction is completed.

These constructions, with the existence of \textit{DQS} of order 33 given in Example 5 and the existence of \textit{PDQS} given in Example 6, permit to determine the spectrum of \textit{DQSs} and \textit{PDQSs}.

\section{5 50 Open research problems}

\textbf{Octagon quadrangle systems OQSs:}

1.) In \([2,3,4]\) the spectrum of \textit{octagon quadrangle systems} \textit{OQSs} and the spectrum of \textit{perfect OQSs} are determined. Examine the possible embedding of a \(C_4\)-design of index \( \mu = 1 \) into an \textit{OQS} of index \( \lambda = 1 \), as collection of \textit{inside} \(C_4\)-cycles.

2.) Determine a relation between the order \( n \) of a \(C_4\)-design embedded into an \textit{OQS} and the order \( v \) of the \textit{OQS}.

3.) Determine the minimum value of \( v \) so that there exists a \(C_4\)-design of order \( n \) and index 1, embedded in an \textit{OQS} of order \( v \) and index 1.

4.) Examine the same problems described in the previous last points, under the condition \( 1 < \mu \leq \lambda \) or also \( \mu = 1 < \lambda \).
5.) The necessary condition so that a $C_4$-design, of order $v$ and index $\mu$, can be nested into an $OQS$, of order $v$ and index $\lambda$, is that $2 \cdot \lambda = 5 \cdot \mu$. Is it true that every $C_4$-design, of order $v$ and index $\mu = 2$, can be nested into an $OQS$ of order $v$ and index $\lambda = 5$?

6.) Consider the same problem in 5) for $\mu = 4$ and $\lambda = 10$. Observe that, for these values, the correspondent spectrum is the largest possible.

7.) In [3][15], upper $C_4$-perfect $OQS$s, lower $C_4$-perfect $HQS$s and superperfect $OQS$s as are considered. Examine the possible embedding of a $C_4$-design of index $\mu = 1$ into an $OQS$ of index $\lambda = 1$, as collection of lateral $C_4$-cycles, where lateral means that $C_4$-cycles are upper or lower.

8.) Determine a relation between the order $n$ of a $C_4$-design lateral-embedded into an $OQS$ and the order $v$ of the $OQS$.

9.) Determine the minimum value of $v$ such that there exists a $C_4$-design of order $n$ and index $\mu = 1$, lateral-embedded into an $OQS$ of order $v$ and index $\lambda = 1$.

10.) Examine the same problems described in the previous last points, under the condition $1 < \mu \leq \lambda$ or also $\mu = 1 < \lambda$.

11.) Is it true that every $C_4$-design, of order $v$ and index $\mu = 2$, can be nested as lateral-$C_4$-design, into an $OQS$ of order $v$ and index $\lambda = 5$?

12.) Consider the same problem in 11) for $\mu = 4$ and $\lambda = 10$.

**Hexagon quadrangle systems HQSs:**

13.) In [12] the spectrum of $HQS$s and the spectrum of perfect $HQS$s have been determined. Examine the possible embedding of a $C_4$-design of index $\mu = 1$ into an $HQS$ of index $\lambda = 1$, as collection of inside $C_4$-cycles.

14.) Determine a relation between the order $n$ of a $C_4$-design embedded into an $HQS$ and the order $v$ of the $HQS$.

15.) Determine the minimum value of $v$ so that there exists a $C_4$-design of order $n$ and index $\mu = 1$, embedded in an $HQS$ of order $v$ and index $\lambda = 1$.

16.) Examine the same problems described in the previous last points, under the condition $1 < \mu \leq \lambda$ or also $\mu = 1 < \lambda$. 
17.) The necessary condition so that a \( C_4 \)-design, of order \( v \) and index \( \mu \), can be nested into an HQS, of order \( v \) and index \( \lambda \), is that \( \lambda = 2 \cdot \mu \). Is it true that every \( C_4 \)-design, of order \( v \) and index \( \mu = 1 \), can be nested into an HQS of order \( v \) and index \( \lambda = 2 \) ?

18.) Consider the same problem of 17) for: \((\lambda, \mu) = (4, 2), (\lambda, \mu) = (6, 3)\) and for \((\lambda, \mu) = (8, 4)\). Observe that, in this last case, the correspondent spectrum is the largest possible.

19.) Define upper \( C_3 \)-perfect HQSs, lower \( C_3 \)-perfect HQSs and superperfect HQSs as in \([3][15]\). Examine the possible embedding of a \( C_3 \)-design of index \( \mu = 1 \) into an HQS of index \( \lambda = 1 \), as collection of lateral \( C_3 \)-cycles, where lateral means that \( C_3 \)-cycles are upper or lower.

20.) Consider lateral \( C_3 \)-perfect HQSs, as in section 4.2 and \([14]\), and determine the spectrum.

21.) Examine the possible embedding of a \( C_3 \)-design of index 1 in an HQS of index 1, as collection of lateral \( C_3 \)-cycles.

22.) Determine a relation between the order \( n \) of a \( C_3 \)-design embedded into an HQS and the order \( v \) of the HQS.

23.) Determine the minimum value of \( v \) such that there exists a \( C_3 \)-design of order \( n \) and index 1, embedded in an HQS of order \( v \) and index 1.

24.) Examine the same problems described in the previous last points, under the condition \( 1 < \mu \leq \lambda \) or also \( \mu = 1 < \lambda \).

25.) The necessary condition so that a \( C_3 \)-design, of order \( v \) and index \( \mu \), can be nested into an HQS, of order \( v \) and index \( \lambda \), is that \( 3 \cdot \lambda = 8 \cdot \mu \). Is it true that every \( C_3 \)-design, of order \( v \) and index 3, can be nested as lateral \( C_3 \)-design, into an HQS of order \( v \) and index 8 ?

Dodecagon quadrangle systems DQSs:

26.) In \([14]\) the spectrum of dodecagon quadrangle systems DQSs and perfect DQSs is determined. Examine the possible embedding of a \( C_4 \)-design of index 1 in a DQS of index 1, as collection of inside \( C_4 \)-cycles.

27.) Determine a relation between the order \( n \) of a \( C_4 \)-design embedded into a DQS and the order \( v \) of the DQS.
28.) Determine the minimum value of \( v \) such that there exists a \( C_4 \)-design of order \( n \) and index 1, embedded in a \( DQS \) of order \( v \) and index 1.

29.) Examine the same problems described in the previous last points, under the condition \( 1 < \mu \leq \lambda \) or also \( \mu = 1 < \lambda \).

30.) Is it true that every \( C_4 \)-design, of order \( v \) and index 1, can be nested into a perfect \( DQS \) of order \( v \) and index 4?

Remark: Observe that for hexagon triple systems \( HTS \) [18] this is true. Indeed, Lucia Gionfriddo proved that every \( C_3 \)-design, of order \( v \) and index 1, can be nested into a perfect \( HTS \) of order \( v \) and index 3?

31.) Define lateral \( C_4 \)-perfect \( DQSs \) and super-perfect \( DQSs \) as in [15], where lateral means that \( C_4 \)-cycles are not inside. Examine the spectrum in the various cases.

32.) Examine the possible embedding of a \( C_4 \)-design of index \( \mu = 1 \) into a \( DQS \) of index \( \lambda = 1 \), as collection of lateral \( C_4 \)-cycles.

33.) Determine a relation between the order \( n \) of a \( C_4 \)-design lateral-embedded into a \( DQS \) and the order \( v \) of the \( DQS \).

34.) Determine the minimum value of \( v \) such that there exists a \( C_4 \)-design of order \( n \) and index 1, lateral-embedded into a \( DQS \) of order \( v \) and index 1.

35.) Examine the same problems described in the previous last points, under the condition \( 1 < \mu \leq \lambda \) or also \( \mu = 1 < \lambda \).

36.) The necessary condition so that a \( C_4 \)-design, of order \( v \) and index \( \mu \), can be nested into a \( DQS \), of order \( v \) and index \( \lambda \), is \( \lambda = 4 \cdot \mu \). Is it true that every \( C_4 \)-design, of order \( v \) and index 1, can be nested as lateral \( C_4 \)-design, into a \( DQS \) of order \( v \) and index 4?

37.) Consider the same problem of 37) for \( \mu = 2 \) and \( \lambda = 8 \), \( \mu = 3 \) and \( \lambda = 12 \) and for \( \mu = 4 \) and \( \lambda = 16 \). Observe that, in this last case, the correspondent spectrum is the largest possible.

**L-Nesting and LG-Nesting:**

38.) In [9] \( L \)-nesting of \( P_k \)-designs having order a prime number has been studied. Examine the existence of \( L \)-nesting of \( P_k \)-designs having order admissible values of \( v \), in the case that \( v \) is not a prime number.
39.) In [10] the spectrum for L-nesting of G-designs, where G has four non-isolated vertices or less, has been determined. Examine the case in which G with more vertices.

40.) Examine the spectrum for L-nesting of C₄-designs.

41.) Examine the spectrum for L-nesting of C_k-designs, for a given k.

42.) Examine the spectrum for nesting of already nested G-designs for G different from paths P_k.

43.) Examine the possible existence of LG-nesting of P_k-designs for admissible values of n in the case that n is not a prime number.

44.) Examine the spectrum for LG-nesting, starting with cycle C_k.

45.) Examine the spectrum for LG-nesting for graph G different from paths and cycles.

46.) Examine what happens for L-nesting of (K₄ - e)-designs when the order v is an even number.

47.) Examine what happens for L-nesting of (K₄ - e)-designs when the order v is not a prime number.

48.) Examine the existence of L-nesting of P₄-designs in the case that the order is one of the following open cases: 10, 12, 14, 16, 20, 22, 28, 34.

49.) For λ ≡ 3 mod 6, µ ≡ 4 mod 8, examine the possible existence of L-nesting of S₃-designs in the case that the order v is one of the following open cases: v=15,27,39,75,87,135,183,195.

50.) For λ ≡ 0 mod 6, µ ≡ 0 mod 8, examine the possible existence of L-nesting of S₃-designs in the case that the order v is one of the following open cases: v = 10, 12, 14, 16, 18, 20, 22, 24, 26, 27, 28, 30, 32, 33, 34, 38, 39, 42, 44, 46, 52, 60, 94, 96, 98, 100, 102, 104, 106, 108, 110, 116, 138, 140, 142, 146, 150, 154, 156, 158, 162, 166, 170, 172, 174, 206, 228.
References


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