Restrained Domination in Graphs
Under Some Binary Operations

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Abstract

In this paper we characterize the restrained dominating sets in the lexicographic product and cartesian product of two connected graphs. The corresponding upper bounds or exact values of the restrained domination numbers of these graphs are also determined.

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1 Introduction

Let \( G = (V(G), E(G)) \) be a simple graph. A subset \( S \) of \( V(G) \) is a \emph{dominating set} in \( G \) if for every \( v \in V(G) \setminus S \), there exists \( x \in S \) such that \( xv \in E(G) \). \( S \) is a \emph{total dominating set} in \( G \) if for every \( v \in V(G) \), there exists \( x \in S \) such that \( xv \in E(G) \). A dominating set \( S \) is a \emph{restrained dominating set} if for every \( v \in V(G) \setminus S \), there exists \( z \in V(G) \setminus S \) such that \( xz \in E(G) \). The \emph{domination number} (resp. \emph{total domination number} and \emph{restrained domination number}) \( \gamma(G) \) (resp. \( \gamma_t(G) \) and \( \gamma^r(G) \)) of \( G \) is the smallest cardinality of a dominating (resp. total dominating and restrained dominating) set in \( G \). A dominating

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set $S$ in $G$ is called a minimum dominating (resp. minimum total dominating, minimum restrained dominating) set in $G$ if the cardinality of $S$ is equal to $\gamma(G)$ (resp. $\gamma_t(G)$ and $\gamma^r(G)$).

As mentioned in [2], an application of the concept of restrained domination is that of prisoners and guards. Each vertex not in the restrained dominating set can be viewed as a position of a prisoner, and every vertex in the restrained dominating set can be considered as a position of a guard. In this way, each prisoner’s position is observed by a guard’s position (to ensure security) while each prisoner’s position is seen by at least one other prisoner’s position (to protect the rights of prisoners). To lessen cost in this situation, it is desirable to place as few guards as possible in a given system (modelled by a graph).

The concepts of restrained dominating sets and restrained domination number were studied and investigated in [1] and [2]. Other types of domination in graphs can be found in [3] and [4].

2 Restrained Domination in the Lexico-
graphic Product of Two Connected Graphs

Recall that the lexicographic product $G[H]$ of two graphs $G$ and $H$ is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u, u')(v, v') \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $u'v' \in E(H)$.

Observe that any subset $C$ of $V(G) \times V(H)$ (in fact, any set of ordered-pairs) can be written as $C = \bigcup_{x \in S} (\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$. Henceforth, we shall use this form to denote any subset $C$ of $V(G) \times V(H)$.

**Theorem 2.1** Let $G$ and $H$ be connected non-trivial graphs. Then a subset $C = \bigcup_{x \in S} (\{x\} \times T_x)$ of $V(G[H])$ is a restrained dominating set in $G[H]$ if and only if $S$ is a dominating set in $G$ possessing the property that

(i) for every $x \in S \setminus N_G(V(G) \setminus S)$ with $T_x \neq V(H)$, either $(V(H) \setminus T_x)$ has no isolated vertices or there exists $y \in S \cap N_G(x)$ with $T_y \neq V(H)$, and

(ii) $T_x$ is a dominating set in $H$ for every $x \in S \setminus N_G(S)$.

**Proof:** Suppose $C$ is a restrained dominating set in $G[H]$. Let $u \in V(G) \setminus S$ and pick $b \in V(H)$. Since $(u, b) \notin C$ and $C$ is a dominating set in $G[H]$, there exists $(y, c) \in C$ such that $(y, c)(u, b) \in E(G[H])$. This implies that $y \in S$ and $uy \in E(G)$. Thus $S$ is a dominating set in $G$.

Now let $x \in S \setminus N_G(V(G) \setminus S)$ with $T_x \neq V(H)$. Let $a \in V(H) \setminus T_x$. Then $(x, a) \notin C$. Since $C$ is a restrained dominating set, there exists $(z, d) \in V(G[H]) \setminus C$ such that $(x, a)(z, d) \in E(G[H])$. Hence, $xz \in E(G)$ or $x = z$. 

and \( ad \in E(H) \). Suppose \( xz \in E(G) \). Since \( x \notin N_G(V(G) \setminus S) \), it follows that \( z \in S \cap N_G(x) \) and \( d \in V(H) \setminus T_z \). Thus \( T_z \neq V(H) \). Suppose \( x = z \). Then \( a, d \in V(H) \setminus T_x \) and \( ad \in E(H) \). This implies that \( a \) is not an isolated vertex in \( \langle V(H) \setminus T_x \rangle \). Therefore, (i) holds.

Next, let \( x \in S \setminus N_G(S) \) and let \( b \in V(H) \setminus T_x \). Then \( (x, b) \notin C \). Since \( C \) is a dominating set, there exists \( (y, c) \in C \) such that \( (x, b)(y, c) \in E(G[H]) \). Since \( y \in S \) and \( x \notin N_G(S) \), it follows that \( x = y \). Therefore, \( c \in T_x \) and \( bc \in E(H) \). This shows that \( T_x \) is a dominating set in \( H \).

For the converse, let \( C = \cup_{x \in S}(\{x\} \times T_x) \) and \( (u, t) \in V(G[H]) \setminus C \). Consider the following cases:

**Case 1.** Suppose that \( u \notin S \).

Since \( S \) is a dominating set of \( G \), there exists \( y \in S \) such that \( uy \in E(G) \). Pick \( a \in T_y \). Then \( (y, a) \in C \) and \( (u, t)(y, a) \in E(G[H]) \). Now, let \( q \in V(H) \setminus \{t\} \) such that \( qt \in E(H) \). Then \( (u, q) \in V(G[H]) \setminus C \) and \( (u, t)(u, q) \in E(G[H]) \).

**Case 2.** Suppose that \( u \in S \).

Consider the following subcases:

**Subcase 1.** Suppose that \( u \in N_G(z) \) for some \( z \in S \setminus \{u\} \).

Then there exists \( (z, b) \in C \) such that \( (u, t)(z, b) \in E(G[H]) \). Further, if \( u \in N_G(v) \) for some \( v \notin S \), then \( (v, t) \notin C \) and \( (u, t)(v, t) \in E(G[H]) \). So suppose \( u \notin N_G(V(G) \setminus S) \). If there exists \( y \in S \cap N_G(u) \) with \( T_y \neq V(H) \), then \( (y, d) \notin C \) for \( d \in V(H) \setminus T_y \) and \( (u, t)(y, d) \in E(G[H]) \). If \( T_y = V(H) \) for all \( y \in S \cap N_G(u) \), then by assumption, \( \langle V(H) \setminus T_u \rangle \) has no isolated vertices. Hence, since \( t \in V(H) \setminus T_u \), there exists \( j \in V(H) \setminus T_u \) such that \( tj \in E(H) \). Therefore, \( (u, j) \notin C \) and \( (u, t)(u, j) \in E(G[H]) \).

**Subcase 2.** Suppose that \( u \notin N_G(z) \) for all \( z \in S \setminus \{u\} \).

Then by assumption, \( T_u \) is a dominating set in \( H \). Since \( (u, t) \notin C \), \( t \notin T_u \). This implies that there exists \( s \in T_u \) such that \( ts \in E(H) \). It follows that \( (u, s) \in C \) and \( (u, t)(u, s) \in E(G[H]) \).

Finally, choose \( w \notin S \) such that \( wu \in E(G) \). Then \( (w, t) \notin C \) and \( (u, t)(w, t) \in E(G[H]) \).

Accordingly, \( C \) is a restrained dominating set in \( G[H] \). \( \square \)

The following remark will be useful.

**Remark 2.2** Let \( G \) be a connected graph and \( S \) a dominating set of \( G \). Then \( \gamma_t(G) \leq |S \cap N_G(S)| + 2|S \setminus N_G(S)| \).

To see this, let \( S \) be a dominating set in \( G \). If \( S \) is a total dominating set, then we are done. So suppose \( S \) is not a total dominating set. Then \( S \setminus N_G(S) \neq \emptyset \). For each \( y \in S \setminus N_G(S) \), choose \( v_y \in V(G) \) such that
Proof. Let \( S \) be a minimum total dominating set of \( G \). Pick \( a \in V(H) \) and set \( T_x = \{a\} \) for every \( x \in S \). Then \( T_x \neq V(H) \) for every \( x \in S \). By Theorem 2.1, \( C = \bigcup_{x \in S} \{x\} \times T_x \) is a restrained dominating set in \( G[H] \). Thus
\[
\gamma^r(G[H]) \leq |C| = \sum_{x \in S} |\{x\} \times T_x| = |S| = \gamma_t(G).
\]

On the other hand, if \( C = \bigcup_{x \in S} \{x\} \times T_x \) is a minimum restrained dominating set in \( G[H] \), then \( S \) is a dominating set in \( G \) by Theorem 2.1. Hence, by Remark 2.2 and Theorem 2.1,
\[
\gamma^r(G[H]) = |C| = \sum_{x \in S} |\{x\} \times T_x| + \sum_{x \in S \setminus N_G(S)} |\{x\} \times T_x| \\
\geq |S \cap N_G(S)| + 2|S \setminus N_G(S)| \geq \gamma_t(G).
\]

Therefore, \( \gamma^r(G[H]) = \gamma_t(G) \). \( \Box \)

**Theorem 2.4** Let \( G \) be a connected graph and \( H \) a non-trivial connected graph with \( \gamma(H) = 1 \). Then \( \gamma^r(G[H]) = \gamma(G) \).

Proof. Let \( S \) be a minimum dominating set of \( G \). Pick \( a \in V(H) \) such that \( \{a\} \) is a dominating set in \( H \) and set \( T_x = \{a\} \) for every \( x \in S \). Clearly, \( T_x \) is a dominating set in \( H \) for every \( x \in S \). Also, if \( x \in S \setminus N_G(V(G) \setminus S) \), then there exists \( y \in S \setminus N_G(x) \) with \( T_y = \{a\} \neq V(H) \). Thus, by Theorem 2.1, \( C = \bigcup_{x \in S} \{x\} \times T_x \) is a restrained dominating set in \( G[H] \). It follows that
\[
\gamma^r(G[H]) \leq |C| = \sum_{x \in S} |\{x\} \times T_x| = |S| = \gamma(G).
\]

Now, if \( C = \bigcup_{x \in S} \{x\} \times T_x \) is a minimum restrained dominating set in \( G[H] \), then \( S \) is a dominating set in \( G \) by Theorem 2.1 and \( T_x \) is a dominating set in \( H \) for every \( x \in S \). Thus
\[
\gamma^r(G[H]) = |C| = \sum_{x \in S \cap N_G(S)} |\{x\} \times T_x| + \sum_{x \in S \setminus N_G(S)} |\{x\} \times T_x| \\
\geq |S \cap N_G(S)| + |S \setminus N_G(S)| = |S| \geq \gamma(G).
\]

Therefore, \( \gamma^r(G[H]) = \gamma(G) \). \( \Box \)

As a quick consequence, we have
Corollary 2.5 Let $G$ be a connected graph and $K_n$ the complete graph of order $n \geq 2$. Then $\gamma^r(G[K_n]) = \gamma(G)$.

3 Restrained Domination in the Cartesian Product of Two Connected Graphs

The Cartesian product $G \times H$ of two graphs $G$ and $H$ is the graph with $V(G \times H) = V(G) \times V(H)$ and $(u, u')(v, v') \in E(G \times H)$ if and only if either $uv \in E(G)$ and $u' = v'$ or $u = v$ and $u'v' \in E(H)$.

Theorem 3.1 Let $G$ and $H$ be non-trivial connected graphs. Then a subset $C = \bigcup_{x \in S} \{x\} \times T_x$ of $V(G \times H)$ is a restrained dominating set in $G \times H$ if and only if $S$ is a dominating set in $G$ possessing the property that

(i) $T_x$ is a dominating set in $H$ for every $x \in S \setminus N_G(S)$;

(ii) For each $x \in S \cap N_G(S)$ and for each $a \in V(H) \setminus T_x$, there exists $b \in T_x \cap N_H(a)$ or there exists $y \in S \cap N_G(x)$ with $a \in T_y$;

(iii) For each $x \in S \setminus N_G(V(G) \setminus S)$ and each $t \in V(H) \setminus T_x$, there exists $d \in [V(H) \setminus T_x] \cap N_H(t)$ or there exists $z \in S \cap N_G(x)$ with $t \notin V(H) \setminus T_x$; and

(iv) $\bigcup \{T_y : y \in S \cap N_G(x)\} = V(H)$ for all $x \notin S$.

Proof: Suppose $C = \bigcup_{x \in S} \{x\} \times T_x$ of $V(G \times H)$ is a restrained dominating set in $G \times H$. Let $u \in V(G) \setminus S$ and pick $b \in V(H)$. Since $(u, b) \notin C$ and $C$ is a dominating set in $G \times H$, there exists $(y, c) \in C$ such that $(y, c)(u, b) \in E(G \times H)$. Since $y \neq u$, $c = b$ and $uy \in E(G)$, where $y \in S$. This implies that $S$ is a dominating set in $G$.

Now let $x \in S \setminus N_G(S)$ and let $a \in V(H) \setminus T_x$. Then $(x, a) \notin C$. Since $C$ is a dominating set, there exists $(y, c) \in C$ such that $(x, a)(y, c) \in E(G \times H)$. Since $y \in S$ and $x \notin N_G(S)$, it follows that $x = y$. Therefore, $c \in T_x$ and $ac \in E(H)$. This shows that $T_x$ is a dominating set in $H$; hence (i) holds.

Let $x \in S \cap N_G(S)$ and let $a \in V(H) \setminus T_x$. Then $(x, a) \notin C$. Since $C$ is dominating, there exists $(y, b) \in C$ such that $(x, a)(y, b) \in E(G \times H)$. This implies that $x = y$ and $ab \in E(H)$ or $xy \in E(G)$ and $a = b$. If $x = y$ and $ab \in E(H)$, then $b \in T_x \cap N_H(a)$. If $xy \in E(G)$ and $a = b$, then $y \in S \cap N_G(x)$ with $a = b \in T_y$. This shows that (ii) holds.

Next, let $x \in S \setminus N_G(V(G) \setminus S)$ and let $t \in V(H) \setminus T_x$. Then $(x, t) \notin C$. Since $C$ is a restrained dominating set, there exists $(z, d) \in V(G \times H) \setminus C$ such that $(x, t)(z, d) \in E(G \times H)$. Hence, $xz \in E(G)$ and $t = d$ or $x = z$ and $td \in E(H)$. If $x = z$ and $td \in E(H)$, then $d \in [V(H) \setminus T_x] \cap N_H(t)$. Suppose $xz \in E(G)$ and...
Thus, condition (iii) holds.

Finally, let \( x \notin S \) and let \( h \in V(H) \). Then \( (x,h) \notin C \). Since \( C \) is a dominating set, there exists \( (z,p) \in C \) such that \( (x,h)(z,p) \in E(G \times H) \). Hence, \( z \in S \cap N_G(x) \) and \( p = h \in T_z \). It follows that \( \cup \{T_y : y \in S \cap N_G(x) \} = V(H) \). This shows that (iv) also holds.

For the converse, let \( C = \cup_{x \in S} \{x\} \times T_x \) and suppose that \( S \) is a dominating set satisfying conditions (i), (ii),(iii), and (iv). Let \( (u,v) \in V(G \times H) \setminus C \).

Consider the following cases:

**Case 1.** Suppose that \( u \notin S \).

By (iv), there exists \( y \in S \cap N_G(u) \) such that \( v \in T_y \). Thus \( (y,v) \in C \) and \( (u,v)(y,v) \in E(G \times H) \). Now, choose \( q \in V(H) \) such that \( vq \in E(H) \). Then \( (u,q) \notin C \) and \( (u,v)(u,q) \in E(G \times H) \).

**Case 2.** Suppose that \( u \in S \). Consider the following subcases:

**Subcase 1.** Suppose that \( u \in N_G(z) \) for some \( z \in S \setminus \{u\} \).

By (ii), there exists \( b \in T_u \cap N_H(v) \) or there exists \( y \in S \cap N_G(u) \) with \( v \in T_y \). In the former \( (u,b) \in C \) and \( (u,v)(u,b) \in E(G \times H) \) and in the latter, \( (y,v) \in C \) and \( (u,v)(y,v) \in E(G \times H) \). Further, if \( u \in N_G(w) \) for some \( w \notin S \), then \( (w,v) \notin C \) and \( (u,v)(w,v) \in E(G \times H) \). So suppose \( u \notin N_G(V(G) \setminus S) \).

By (iii), there exists \( d \in [V(H) \setminus T_u] \cap N_H(v) \) or there exists \( z \in S \cap N_G(u) \) with \( v \notin V(H) \setminus T_z \). Then \( (u,d) \notin C \) and \( (u,v)(u,d) \in E(G \times H) \) or \( (z,v) \notin C \) and \( (u,v)(z,v) \in E(G \times H) \).

**Subcase 2.** Suppose that \( u \notin N_G(z) \) for all \( z \in S \setminus \{u\} \).

By (i), \( T_u \) is a dominating set in \( H \). Since \( (u,v) \notin C \), \( v \notin T_u \). This implies that there exists \( s \in T_u \) such that \( vs \in E(H) \). It follows that \( (u,s) \in C \) and \( (u,v)(u,s) \in E(G \times H) \).

Finally, choose \( w \notin S \) such that \( wu \in E(G) \). Then \( (w,v) \notin C \) and \( (u,v)(w,v) \in E(G \times H) \).

Accordingly, \( C \) is a restrained dominating set in \( G \times H \).

**Corollary 3.2** Let \( G \) and \( H \) be connected non-trivial graphs. Then \( \gamma^r(G \times H) \leq \min\{|V(G)|\gamma^r(H),|V(H)|\gamma^r(G)\} \).

**Proof:** Let \( S \) be a minimum restrained dominating set of \( G \). For each \( x \in S \), set \( T_x = V(H) \). Then the set \( C = \cup_{x \in S} \{x\} \times T_x \) is a minimum restrained dominating set in \( G \times H \). Thus, by Theorem 3.1, \( \gamma^r(G \times H) \leq |C| = |V(G)|\gamma^r(H) \). Interchanging the roles of \( G \) and \( H \), we obtain the desired result.
References


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