Cycle Derivatives of the Generalized Fan
and the Generalized Wheel

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Abstract

The cycle derivative of a graph $G$, denoted by $G'$, is a graph obtained from $G$ by treating the prime cycles (also called chordless cycles or induced cycles) of $G$ as vertices of $G'$ and two vertices $x$ and $y$ in $G'$ are adjacent if and only if the corresponding cycles $C(x)$ and $C(y)$ have a common edge in $G$.

This study establishes the cycle derivatives of the generalized fan and the generalized wheel. Also the sizes of the cycle derivatives of these graphs are determined.

Mathematics Subject Classification: 05C30

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1 Introduction

Let $G$ be a connected cyclic graph. An edge in a graph $G$ joining two non-consecutive vertices of a cycle is a chord of the cycle. A cycle in a graph $G$ is called a prime cycle if it is chordless. The cycle derivative of a cyclic graph $G$, denoted by $G'$, is a graph obtained from $G$ whose vertices correspond to all the primes cycles determined in $G$, and where two vertices in $G'$ are adjacent if and only if the corresponding cycles have a common edge in $G$.

We first introduce the following definition of the join of graphs since the generalized fan and the generalized wheel are expressed as the join of graphs. We also include here the definition of the Cartesian product of graphs since our results involve this graph operation.

Definition 1.1 [3] Let $G_1$ and $G_2$ be vertex disjoint graphs. The join $G_1 + G_2$ of $G_1$ and $G_2$ has vertex-set $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and edge-set $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{[u,v] : u \in V(G_1), v \in V(G_2)\}$.

Definition 1.2 [3] The Cartesian product $G \times H$ of two graphs $G$ and $H$ is the graph with vertex set $V(G \times H) = V(G) \times V(H)$ and edge set $E(G \times H)$ satisfying the following condition: $(u, u')(v, v') \in E(G \times H)$ if and only if either $u = v$ and $u'v' \in E(H)$ or $u' = v'$ and $uv \in E(G)$.

2 Cycle Derivative of the Generalized Fan

This section establishes the cycle derivative of the generalized fan $F_{m,n} = \overline{K_m} + P_n$. For convenience, we use $\overline{K_m} + P_n$ for the generalized fan $F_{m,n}$ to ease our approach in the proof.

Lemma 2.1 For $m \geq 1$ and $n \geq 2$, $\overline{K_m} + P_n$ has $m(n-1)$ prime cycles.

Proof: Let $V(\overline{K_m}) = \{v_1, v_2, v_3, \ldots, v_m\}$ and $P_n = [x_1, x_2, x_3, \ldots, x_n]$. Then for every pair of vertices $x_r$ and $x_{r+1}$ in $P_n$, there are $m$ primes cycles formed. These prime cycles are triangles of the form $[x_r, x_{r+1}, v_i, x_r]$, for all $r = 1, 2, \ldots, n-1$ and for all $i = 1, 2, \ldots, m$. Since there are $(n-1)$ such pairings of vertices, then there are $m(n-1)$ prime cycles formed. \Box

Theorem 2.2 Let $m \geq 1$ and $n \geq 2$. Then

$$(\overline{K_m} + P_n)' = \begin{cases} P_{n-1} & , \text{ } m = 1 \text{ } \text{ and } \text{ } n > 1 \\ K_m & , \text{ } m \geq 2 \text{ } \text{ and } \text{ } n = 2 \\ K_m \times P_{n-1} & , \text{ } m \geq 2 \text{ } \text{ and } \text{ } n \geq 3 \end{cases}$$

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Proof: Let $V \left( \overline{K_m} \right) = \{v_1, v_2, \ldots, v_m\}$ and $P_n = [x_1, x_2, x_3, \ldots, x_n]$. Consider the following cases:

**Case 1:** Suppose $m = 1$ and $n > 1$.
Then $\left( \overline{K_1} + P_n \right)' = (F_n)' = P_{n-1}$.

**Case 2:** Suppose $m \geq 2$ and $n = 2$.
Now, $\overline{K_m} + P_2$ has $m$ prime cycles. Thus $\left( \overline{K_m} + P_2 \right)'$ has $m$ vertices. Now, for $V \left( P_2 \right) = \{x_1, x_2\}$, we have the prime cycles of the form $[x_1, x_2, v_i, x_1]$, for all $i = 1, 2, \ldots, m$. Note that $x_1x_2$ is the common edge, so the cycles $[x_1, x_2, v_i, x_1]$ are pairwise adjacent. Hence $\left( \overline{K_m} + P_2 \right)' = K_m$.

**Case 3:** Suppose $m \geq 2$ and $n \geq 3$.
Then the vertices of $\left( \overline{K_m} + P_n \right)'$ are prime cycles of the form $[x_r, x_{r+1}, v_i, x_r]$ for all $r = 1, 2, \ldots, n - 1$ and for all $i = 1, 2, \ldots, m$. Note that $\left( \overline{K_m} + P_n \right)'$ has $m(n - 1)$ vertices by Lemma 2.1. Since there are $n - 1$ pairs of adjacent vertices in $P_n$, then position or write the first $n - 1$ vertices of $\left( \overline{K_m} + P_n \right)'$ in a row formation. Since the edge $x_rx_{r+1}$ is common to the cycles $[x_r, x_{r+1}, v_i, x_1]$ for all $i = 1, 2, \ldots, m$, the first $m$ vertices forms a complete graph $K_m$. Now draw the remaining $(n - 2)$-$K_m$’s. Observe that an edge $v_ix_r$ is common to cycles $[x_{r-1}, x_r, v_i, x_{r-1}]$, for all $r = 2, 3, \ldots, n - 1$. Hence for every pair of prime cycles of the form $[x_{r-1}, x_r, v_i, x_{r-1}]$ and $[x_r, x_{r+1}, v_i, x_r]$ in the same row are adjacent with each other for all $r = 2, 3, \ldots, n - 1$ and for all $i = 1, 2, \ldots, m$. Therefore $\left( \overline{K_m} + P_n \right)' = K_m \times P_{n-1}$. □

Let us now consider counting the number of edges of $(F_{m,n})'$. This gives us ideas of the size of the cycle derivative in relation to the original graph. Our result follows directly from the preceding theorem.

**Theorem 2.3** Let $m \geq 1$ and $n \geq 2$. Then

$$\left| E \left( \overline{K_m} + P_n \right)' \right| = \begin{cases} n - 2, & m = 1 \text{ and } n > 1 \\ \frac{m^2 - m}{2}, & m \geq 2 \text{ and } n = 2 \\ \frac{m^2n - m^2 + mn - 3m}{2}, & m \geq 2 \text{ and } n \geq 3 \end{cases}$$

**Proof:** For $m = 1$ and $n > 1$, by Theorem 2.2, $\left( \overline{K_1} + P_n \right)' = P_{n-1}$. Thus,

$$\left| E \left( \overline{K_m} + P_n \right)' \right| = |E \left( P_{n-1} \right)| = n - 2.$$ 

For $m \geq 2$ and $n = 2$, by Theorem 2.2,

$$\left| E \left( \overline{K_m} + P_2 \right)' \right| = |E \left( K_m \right)|$$ 

$$= \binom{m}{2}$$ 

$$= \frac{m^2 - m}{2}.$$
Now, for \( m \geq 2 \) and \( n \geq 3 \), \( (\overline{K_m} + P_n)' = K_m \times P_{n-1} \). Then,

\[
\left| E \left( (\overline{K_m} + P_n)' \right) \right| = \left| E \left( K_m \times P_{n-1} \right) \right| = |V(K_m)||E(P_{n-1})| + |V(P_{n-1})||E(K_m)|
\]

\[
= m(n-2) + (n-1) \left( \frac{m^2-m}{2} \right)
\]

\[
= mn - 2m + \frac{(n-1)(m^2-m)}{2}
\]

\[
= \frac{2mn - 4m + m^2n - mn - m^2 + m}{2}
\]

\[
= \frac{m^2n - m^2 + mn - 3m}{2}.
\]

This completes the proof. \( \square \)

The next section establishes the results on the cycle derivative of the generalized wheel \( W_{m,n} = \overline{K_m} + C_n \).

3 Cycle Derivative of the Generalized Wheel

First, we count the number of prime cycles of the generalized wheel \( W_{m,n} \). Our result is established in the following lemma.

**Lemma 3.1** For \( m \geq 1 \) and \( n \geq 2 \), \( \overline{K_m} + C_n \) has \( mn + 1 \) prime cycles.

**Proof:** Let \( V(\overline{K_m}) = \{v_1, v_2, \ldots, v_m\} \) and \( C_n = [x_1, x_2, \ldots, x_n] \). For every pair of vertices \( x_r \) and \( x_{r+1} \) in \( C_n \) where \( r = 1, 2, \ldots, n \) and \( n + 1 = 1 \), there are \( m \) prime cycles formed. These prime cycles are triangles of the form \( [x_r, x_{r+1}, v_i, x_r] \), for all \( r = 1, 2, \ldots, n \) and for all \( i = 1, 2, \ldots, m \). Since there are \( n \) pairings of vertices, then there are \( mn \) prime cycles formed. Since \( C_n \) is a prime cycle itself then \( \overline{K_m} + C_n \) has \( mn + 1 \) prime cycles. \( \square \)

Now, we are ready to determine the cycle derivative of the generalized wheel \( W_{m,n} \). The following result gives the explicit form of the derivative of \( W_{m,n} \).

**Theorem 3.2** For \( m \geq 2 \) and \( n \geq 3 \), \( (\overline{K_m} + C_n)' = (K_m \times C_n) + K_1 \).

**Proof:** Let \( V(\overline{K_m}) = \{v_1, v_2, \ldots, v_m\} \) and \( C_n = [x_1, x_2, \ldots, x_n] \). Then by Lemma 2.4, \( \overline{K_m} + C_n \) has \( mn + 1 \) prime cycles. Thus \( (\overline{K_m} + C_n)' \) has \( mn + 1 \) vertices. Note that this number may be obtained as follows. First, work on the \( mn \) vertices of \( (\overline{K_m} + C_n)' \) of the form \( [x_r, x_{r+1}, v_i, x_r] \) and \( [x_n, x_1, v_i, x_n] \) where \( r \leq n-1 \) and \( i \leq m \). Choose a pair of vertices \( x_r \) and \( x_{r+1} \) and then position or
write the first \( m \) vertices of \((K_m + C_n)'\) in a column formation. Since the edge \( x_r x_{r+1} \) is common to the \( m \) vertices, i.e., the prime cycles, the first \( m \) vertices forms the complete graph \( K_m \). Since there are \( n \) pairs of adjacent vertices in \( C_n \), i.e., \( x_r \) and \( x_{r+1} \), \( x_1 \) and \( x_2 \), then draw the remaining \((n - 1) K_m\)'s. Observe that an edge \( v_r \) is common to the prime cycles \([x_{r-1}, x_r, v_i, x_{r-1}]\) while an edge \( v_i x_1 \) is common to the prime cycles \([x_n, x_1, v_i, x_n]\). Hence every pair of vertices of the form \([x_{r-1}, x_r, v_i, x_{r-1}]\) and \([x_n, x_1, v_i, x_n]\) in the same row are adjacent to each other. Also, \([x_1, x_2, v_i, x_1]\) and \([x_n, x_1, v_i, x_n]\) are adjacent. Thus the graph \( K_m \times C_n \) is formed. Now position the vertex representing the cycle \( z = [x_1, x_2, \ldots, x_n, x_1] \), where \( n \geq 3 \). Note that the edges \( x_r x_{r+1} \) and \( x_n x_1 \) are also edges of the prime cycles \([x_r, x_{r+1}, v_i, x_r]\) and \([x_1, v_i, x_n, x_1]\), respectively, for all \( i = 1, 2, \ldots, m \) and for all \( r = 1, 2, \ldots, n - 1 \). 

Theorem 3.3 Let \( m \geq 1 \) and \( n \geq 3 \). Then

\[
|E((K_m + C_n)')| = \begin{cases} 
2n, & m = 1 \text{ and } n \geq 3 \\
\frac{m^2 n + 3mn}{2}, & m \geq 2 \text{ and } n \geq 3
\end{cases}
\]

Proof: For \( m = 1 \) and \( n \geq 3 \), \((K_1 + C_n)' = W_n\). Then

\[
|E((K_m + C_n)')| = |E(W_n)| = 2n.
\]

For \( m \geq 2 \) and \( n \geq 3 \), \((K_m + C_n)' = (K_m \times C_n) + K_1\), by Theorem 2.5. Thus,

\[
|E((K_m + C_n)')| = |E(K_m \times C_n) + K_1| \\
= |E(K_m \times C_n)| + |E(K_1)| + |V(K_m \times C_n)| |V(K_1)| \\
= |V(K_m)| |E(C_n)| + |V(C_n)| |E(K_m)| \\
+ |E(K_1)| + |V(K_m \times C_n)| |V(K_1)| \\
= mn + n \left( \frac{m^2 - m}{2} \right) + 0 + mn(1) \\
= mn + \frac{m^2 n - mn}{2} + mn \\
= \frac{2mn + m^2 n - mn + 2mn}{2} \\
= \frac{m^2 n + 3mn}{2}.
\]

The result follows. \( \square \)

The above results shows that \(|E(W_m, n)| \leq |E(W'_m, n)|\) and equality holds if and only if \( m = 1 \).
References


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