A Note on Generalized Twisted 
$(h, q)$-Tangent Numbers and Polynomials

C. S. Ryoo

Department of Mathematics
Hannam University, Daejeon 306-791, Korea

Abstract

In this paper we construct the generalized twisted $(h, q)$-tangent numbers $T_{n,\chi,q,w}^{(h)}$ and polynomials $T_{n,\chi,q,w}^{(h)}(x)$. Some interesting results and relationships are obtained.

Mathematics Subject Classification: 11B68, 11S40, 11S80

Keywords: tangent numbers and polynomials, twisted tangent numbers and tangent polynomials, generalized twisted $(h, q)$-tangent numbers and polynomials

1 Introduction

Recently, many mathematicians have studied different kinds of the Euler, Bernoulli, Genocchi, Tangent numbers and polynomials(see [1-6]). These numbers and polynomials play important roles in many different areas of mathematics such as number theory, combinatorics, special function and analysis. The purpose of this paper is to construct the generalized twisted $(h, q)$-tangent polynomials $T_{n,\chi,q,w}^{(h)}(x)$ attached to $\chi$ and derive a new $l$-series which interpolates the generalized twisted $(h, q)$-tangent polynomials $T_{n,\chi,q,w}^{(h)}(x)$. Throughout this paper we use the following notations. By $\mathbb{N}$ we denote the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, the set of integers forms a ring that is denoted $\mathbb{Z}$, and $\mathbb{C}$ denotes the complex number field.
In [3], we introduced the twisted \((h,q)\)-tangent numbers \(T_{n,q,w}^{(h)}\) and polynomials \(T_{n,q,w}^{(h)}(x)\). Let \(q\) be a complex number with \(|q| < 1\), \(h \in \mathbb{Z}\), and \(w\) be the \(p^{N}\)-th root of unity. The twisted \((h,q)\)-tangent numbers \(T_{n,q,w}^{(h)}\) and polynomials \(T_{n,q,w}^{(h)}(x)\) are defined by the following generating functions

\[
2 \frac{e^{2t}}{wq^{he^{2t}} + 1} = \sum_{n=0}^{\infty} T_{n,q,w}^{(h)} \frac{t^{n}}{n!},
\]

and

\[
\left(2 \frac{e^{2t}}{wq^{he^{2t}} + 1}\right) e^{xt} = \sum_{n=0}^{\infty} T_{n,q,w}^{(h)}(x) \frac{t^{n}}{n!}.
\]

Clearly, \(T_{n,q,w}^{(h)}(0) = T_{n,q,w}^{(h)}\). Ryoo[4] introduced the generalized twisted tangent polynomials \(T_{n,\chi,w}^{(h)}(x)\). Let \(\chi\) be Dirichlet’s character with conductor \(d \in \mathbb{N}\) with \(d \equiv 1(mod 2)\). Then the generalized twisted tangent numbers associated with \(\chi\), \(T_{n,\chi,w}^{(h)}\), and twisted tangent polynomials associated with \(\chi\), \(T_{n,\chi,w}^{(h)}(x)\), are defined by the following generating functions

\[
2 \sum_{a=0}^{d-1} \chi(a)(-1)^{a}w^{a} e^{2at} \frac{e^{2t}}{w^{d}e^{2dt} + 1} = \sum_{n=0}^{\infty} T_{n,\chi,w}^{(h)}(x) \frac{t^{n}}{n!},
\]

and

\[
\left( 2 \sum_{a=0}^{d-1} \chi(a)(-1)^{a}w^{a} e^{2at} \right) e^{xt} \frac{e^{2t}}{w^{d}e^{2dt} + 1} = \sum_{n=0}^{\infty} T_{n,\chi,w}^{(h)}(x) \frac{t^{n}}{n!}.
\]

\[\text{(2.1)}\]

2 Generalized twisted \((h,q)\)-tangent polynomials

Our primary aim in this section is to define generalized twisted \((h,q)\)-tangent numbers and polynomials. These numbers will be used to prove the analytic continuation of the \(l\)-series. Let \(q\) be a complex number with \(|q| < 1\), \(h \in \mathbb{Z}\), and \(w\) be the \(p^{N}\)-th root of unity. Let \(\chi\) be Dirichlet’s character with conductor \(d \in \mathbb{N}\) with \(d \equiv 1(mod 2)\). Then the generalized twisted \((h,q)\)-tangent numbers associated with \(\chi\), \(T_{n,\chi,q,w}^{(h)}\), are defined by the following generating function

\[
F_{\chi,q,w}^{(h)}(t) = 2 \sum_{a=0}^{d-1} \chi(a)(-1)^{a}w^{a}q^{ha} e^{2at} \frac{e^{2t}}{w^{d}q^{hd}e^{2dt} + 1} = \sum_{n=0}^{\infty} T_{n,\chi,q,w}^{(h)} \frac{t^{n}}{n!}.
\]

We now consider the generalized twisted \((h,q)\)-tangent polynomials associated with \(\chi\), \(T_{n,\chi,q,w}^{(h)}(x)\), are also defined by

\[
F_{\chi,q,w}^{(h)}(x, t) = \left( 2 \sum_{a=0}^{d-1} \chi(a)(-1)^{a}w^{a}q^{ha} e^{2at} \right) e^{xt} \frac{e^{2t}}{w^{d}q^{hd}e^{2dt} + 1} = \sum_{n=0}^{\infty} T_{n,\chi,q,w}^{(h)}(x) \frac{t^{n}}{n!}.
\]

\[\text{(2.2)}\]
When \( \chi = \chi^0 \), above (2.1) and (2.2) will become the corresponding definitions of the twisted \((h, q)\)-tangent numbers \( T_{n,q,w}^{(h)} \) and polynomials \( T_{n,q,w}(x) \) (see [3]). If \( q \to 1 \), above (2.1) and (2.2) will become the corresponding definitions of the generalized twisted tangent numbers \( T_{n,\chi,w} \) and polynomials \( T_{n,\chi,w}(x) \) (see [4]).

It is easy to deduce that generalized twisted \((h, q)\)-tangent polynomials associated with \( \chi \), \( T_{n,\chi,q,w}^{(h)}(x) \) satisfy

\[
2 \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a q^h a e^{2at} \frac{e^{xt}}{w^d q^h e^{2dt} + 1} = \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a q^{ha} \left( \frac{2e^{(2a+x)dt}}{w^d q^h e^{2dt} + 1} \right) = \sum_{m=0}^{\infty} \left( d^m \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a q^{ha} T_{m,q,w}^{(h)} \left( \frac{2a + x}{d} \right) \right) \frac{t^m}{m!}.
\]

Thus we have the following theorem.

**Theorem 2.1** Let \( \chi \) be Dirichlet’s character with conductor \( d \in \mathbb{N} \) with \( d \equiv 1(\text{mod} 2) \). Then we have

1. \( T_{n,\chi,q,w}^{(h)}(x) = d^n \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a q^h a T_{m,q,w}^{(h)} \left( \frac{2a + x}{d} \right) \)

2. \( T_{n,\chi,q,w}^{(h)} = d^n \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a q^{ha} T_{m,q,w}^{(h)} \left( \frac{2a}{d} \right) \)

3. \( T_{n,\chi,q,w}^{(h)}(x) = \sum_{l=0}^{n} \left( \frac{n}{l} \right) T_{l,\chi,q,w}^{(h)} e^{n-l} \)

For \( n \in \mathbb{N} \) with \( n \equiv 0(\text{mod} 2) \), we obtain

\[
-2 \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a q^{ha} e^{2at} \frac{e^{xt}}{w^d q^h e^{2dt} + 1} + 2 \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a q^{ha} e^{2at} \frac{e^{xt}}{w^d q^h e^{2dt} + 1} = \sum_{m=0}^{\infty} \left( 2 \sum_{a=0}^{nd-1} \chi(a)(-1)^a w^a q^{ha} (2a)^m \right) \frac{t^m}{m!}.
\]

By comparing coefficients of \( \frac{t^m}{m!} \) in the above equation, we have the following theorem:

**Theorem 2.2** Let \( \chi \) be Dirichlet’s character with conductor \( d \in \mathbb{N} \) with \( d \equiv 1(\text{mod} 2) \), \( n \) a positive even integer, and \( m \in \mathbb{N} \). Then we have

\[
2 \sum_{a=0}^{nd-1} \chi(a)(-1)^a w^a q^{ha} (2a)^m = -w^{nd} q^{hd} T_{m,\chi,q,w}^{(h)}(2nd) + T_{m,\chi,q,w}^{(h)}.
\]
Next, we introduce the $l$-series and two variable $l$-series.

**Definition 2.3** For $s \in \mathbb{C}$, define two variable $l$-series as

$$l_{q,w}^{(h)}(s, x|\chi) = 2 \sum_{m=0}^{\infty} \frac{(-1)^m \chi(m) w^m q^m}{(2m + x)^s}.$$ 

By using (2.2), we easily see that

$$F_{\chi,q,w}^{(h)}(x, t) = \frac{2 \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a q^a e^{2at}}{w^d q^d e^{2dt} + 1} e^{xt}$$

$$= 2 \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a q^a e^{(2a+2)xt} \sum_{l=0}^{\infty} (-1)^l w^l q^l e^{2l dt}$$

$$= 2 \sum_{a=0}^{d-1} \sum_{l=0}^{\infty} \chi(a)(-1)^{a+l} w^a q^l e^{(2a+2l) dt}$$

$$= 2 \sum_{m=0}^{\infty} \chi(m)(-1)^m w^m q^m e^{(2m+x)t}.$$ 

Then we have

$$\left( \frac{d}{dt} \right)^k F_{\chi,q,w}^{(h)}(x, t) \bigg|_{t=0} = 2 \sum_{n=0}^{\infty} \chi(n)(-1)^n w^n q^n(2n + x)^k, \quad (2.3)$$

and

$$\left( \frac{d}{dt} \right)^k \left( \sum_{n=0}^{\infty} T_{n,\chi,q,w}^{(h)}(x) \frac{t^n}{n!} \right) \bigg|_{t=0} = T_{k,\chi,q,w}^{(h)}(x), \quad \text{for } k \in \mathbb{N}. \quad (2.4)$$

By (2.3), (2.4), we have the following theorem.

**Theorem 2.4** For any positive integer $k$, we have

$$T_{k,\chi,q,w}^{(h)}(x) = l_{q,w}^{(h)}(-k, x|\chi).$$

**Definition 2.5** For $s \in \mathbb{C}$, define $l$-series as

$$l_{q,w}^{(h)}(s | \chi) = 2 \sum_{m=1}^{\infty} \frac{(-1)^m \chi(m) w^m q^m}{(2m)^s}.$$ 

By simple calculation, we have the following theorem.

**Theorem 2.6** For any positive integer $k$, we have

$$l_{q,w}^{(h)}(-k | \chi) = T_{k,\chi,q,w}^{(h)}.$$
3 Witt-type formulae on $\mathbb{Z}_p$ in $p$-adic number field

Our primary aim in this section is to obtain the Witt-type formulae of the generalized twisted $(h, q)$-tangent numbers $T_{n, \chi, q, w}^{(h)}$ and polynomials $T_{n, \chi, q, w}(x)$ attached to $\chi$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of extension, $q$ is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.

$$g \in UD(\mathbb{Z}_p) = \{g|g: \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\}.$$ 

Kim[1] defined the fermionic $p$-adic invariant integral on $\mathbb{Z}_p$ as follows:

$$I_{-1}(g) = \int_X g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} g(x)(-1)^x. \quad (3.1)$$

If we take $g_n(x) = g(x+n)$ in (3.1), then we see that

$$I_{-1}(g_n) = (-1)^n I_{-1}(g) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} g(l). \quad (3.2)$$

Let $T_p = \bigcup_{N \geq 1} C_p^N = \lim_{N \to \infty} C_p^N$, where $C_p^N = \{w|w^p = 1\}$ is the cyclic group of order $p^N$. For $w \in T_p$, we denote by $\phi_w: \mathbb{Z}_p \to \mathbb{C}_p$, the locally constant function $x \mapsto w^x$ (see [6]).

We assume that $q \in \mathbb{C}_p$ with $|q - 1|_p < 1$. Let $\chi$ be the primitive Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1$(mod 2), $w \in T_p$, and $h \in \mathbb{Z}$. Let $g(y) = \chi(y)\phi_w(y)q^{hy}e^{(2y+x)t}$. By (3.1), we derive

$$I_1(\chi(y)\phi_w(y)q^{hy}e^{(2y+x)t}) = \int_X \chi(y)\phi_w(y)q^{hy}e^{(2y+x)t} d\mu_{-1}(y)$$

$$= \left(\frac{2 \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a q^{ha} e^{2at}}{w^d q^{hd} e^{2dt} + 1}\right) e^{xt} \quad (3.3)$$

By using Taylor series of $e^{(2y+x)t}$ in the above equation (3.3), we have

$$\sum_{n=0}^{\infty} \left(\int_X \chi(y)\phi_w(y)q^{hy}(2y + x)^n d\mu_{-1}(y)\right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} T_{n, \chi, q, w}^{(h)}(x) \frac{t^n}{n!}.$$
By comparing coefficients of \( \frac{t^n}{n!} \) in the above equation, we have the Witt formula for the generalized twisted \((h, q)\)-tangent polynomials attached to \( \chi \) as follows:

**Theorem 3.1** For positive integers \( n, q \in \mathbb{C}_p \) with \( |q-1|_p < 1 \), and \( w \in T_p \), we have For positive integers \( n, w \in T_p \), and \( h \in \mathbb{Z} \), we have

\[
T_{n,\chi,q,w}^{(h)}(x) = \int_X \chi(y)\phi_w(y)q^{hy}(2y + x)^n d\mu_{-1}(y). \tag{3.4}
\]

Observe that for \( x = 0 \), the equation (3.4) reduces to (3.5).

**Corollary 3.2** For positive integers \( n, q \in \mathbb{C}_p \) with \( |q-1|_p < 1 \), and \( w \in T_p \), one has

\[
T_{n,\chi,q,w}^{(h)} = \int_X \chi(y)\phi_w(y)q^{hy}(2y)^n d\mu_{-1}(y). \tag{3.5}
\]

By (3.2) and (3.3), we obtain the following theorem:

**Theorem 3.3** For \( n \in \mathbb{Z}_+ \), \( w \in T_p \), and \( h \in \mathbb{Z} \), one has

\[
w^{nd}q^{hnd}T_{m,\chi,q,w}^{(h)}(2nd) - (-1)^nT_{m,\chi,q,w}^{(h)} = 2^{m+1} \sum_{l=0}^{nd-1} (-1)^{n-1-l} \chi(l)w^l q^{hl} l^m.
\]

**References**


Received: August 1, 2014